

Spine decomposition and $L \log L$ criterion for superprocesses with non-local branching mechanisms

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Abstract

In this paper, we provide a pathwise spine decomposition for superprocesses with both local and non-local branching mechanisms under a martingale change of measure. This result complements the related results obtained in [19, 27, 32] for superprocesses with purely local branching mechanisms and in [11, 29] for multitype superprocesses. As an application of this decomposition, we obtain necessary/sufficient conditions for the limit of the fundamental martingale to be non-degenerate. In particular, we obtain extinction properties of superprocesses with non-local branching mechanisms as well as a Kesten-Stigum $L \log L$ theorem for the fundamental martingale.

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1 Introduction

The so-called *spine decomposition* for superprocesses was introduced in terms of a semigroup decomposition by Evans [19]. To be more specific, Evans [19] described the semigroup of a superprocess with branching mechanism $\psi(\lambda) = \lambda^2$ under a martingale change of measure in terms of the semigroup of an immortal particle (called the *spine*) and the semigroup of the original superprocess. Since then there has been a lot of interest in finding the spine decomposition for other types of superprocesses due to a variety of applications. For example, Engländer and Kyprianou [17] used a similar semigroup decomposition to establish the L^1 -convergence of martingales for superdiffusions with quadratic branching mechanisms. Later, Kyprianou et al. [27, 28] obtained a pathwise spine decomposition for a one-dimensional super-Brownian motion with spatially-independent local branching mechanism, in which independent copies of the original superprocess immigrate along the path of the immortal particle, and they used this decomposition to establish the L^p -boundedness ($p \in (1, 2]$) of martingales. A similar pathwise decomposition was obtained by Liu et al. [32] for

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a class of superdiffusions on bounded domains with spatially-dependent local branching mechanisms, and it was used to establish a Kesten-Stigum $L \log L$ theorem for the martingale. In the set-up of branching Markov processes, such as branching diffusions and branching random walks, an analogous decomposition has been introduced and used as a tool to analyze branching Markov processes. See, for example, [22] for a brief history of spine approach for branching Markov processes. Until very recently such a spine decomposition for superprocesses was only available for superprocesses with local branching mechanisms. In a recent paper [29], Kyprianou and Palau established a spine decomposition for a multitype continuous-state branching process and used it to study the extinction properties. Concurrently to their work, a similar decomposition has been obtained by Chen et al. [11] for a multitype superdiffusion, and it has been used to establish a Kesten-Stigum $L \log L$ theorem. In both of these papers, only a very special kind of non-local branching mechanisms are considered. The goal of this paper is to close the gap by establishing a pathwise spine decomposition for superprocesses with both local and general non-local branching mechanisms. Our result shows that, for a superprocess with both local and non-local branching, under a martingale change of measure the spine runs as a copy of a conservative process which can be constructed by concatenating copies of subprocess of the h -transform of the original spatial motion via a transfer kernel determined by the non-local branching, and the general nature of the branching mechanism induces three different kinds of immigration—the *continuous*, *discontinuous* and *revival-caused* immigration. The concatenating procedure and revival-caused immigration are consequences of non-local branching, and they do not occur when the branching mechanism is purely local.

The rest of this paper is organized as follows. We start Section 2 with a review of the basic definitions and properties of non-local branching superprocesses. We introduce the Kuznetsov measures in Section 2 which will be used later. In Section 3, we present our main working assumptions and the fundamental martingale. Then Section 4 provides the spine decomposition for superprocesses with non-local branching mechanisms under the martingale change of measure. In Sections 5 and 6 we use the spine decomposition to find sufficient and necessary conditions for the limit of the fundamental martingale to be non-degenerate respectively. In particular, we obtain extinction properties of the non-local branching superprocess as well as a Kesten-Stigum $L \log L$ theorem for the martingale. In the last section, we give some concrete examples to illustrate our results.

2 Preliminary

2.1 Superprocess with non-local branching mechanisms

Throughout this paper we use “ $:=$ ” as a definition. Suppose that E is a Luzin topological space with Borel σ -algebra $\mathcal{B}(E)$ and m is a σ -finite measure on $(E, \mathcal{B}(E))$ with full support. Let $\mathcal{M}(E)$ denote the space of finite Borel measures on E topologized by the weak convergence. Let $\mathcal{M}(E)^0 := \mathcal{M}(E) \setminus \{0\}$ where 0 denotes the null measure on E . Let $E_\partial := E \cup \{\partial\}$ be the one-point compactification of E . Any function f on E will be automatically extended to

E_∂ by setting $f(\partial) = 0$. When μ is a measure on $\mathcal{B}(E)$ and f, g are measurable functions, let $\langle f, \mu \rangle := \int_E f(x) \mu(dx)$ and $(f, g) := \int_E f(x) g(x) m(dx)$ whenever the right hand sides make sense. Sometimes we also write $\mu(f)$ for $\langle f, \mu \rangle$. For a function f on E , $\|f\|_\infty := \sup_{x \in E} |f(x)|$ and $\text{esssup}_{x \in E} f := \inf_{N: m(N)=0} \sup_{x \in E \setminus N} |f(x)|$. Numerical functions f and g on E are said to be m -equivalent ($f = g$ [m] in notation) if $m(\{x \in E : f(x) \neq g(x)\}) = 0$. If $f(x, t)$ is a function on $E \times [0, +\infty)$, we say f is *locally bounded* if $\sup_{t \in [0, T]} \sup_{x \in E} |f(x, t)| < +\infty$ for every $T \in (0, +\infty)$. For a function $f(x, s)$ defined on $E \times [0, +\infty)$ and a number $t \geq 0$, we denote by $f^t(\cdot)$ the function $x \mapsto f(x, t)$. Throughout this paper we use $\mathcal{B}_b(E)$ (respectively, $\mathcal{B}^+(E)$) to denote the space of bounded (respectively, non-negative) measurable functions on $(E, \mathcal{B}(E))$. For $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$, and $\log^+ a := \log(a \vee 1)$.

Let $\xi = (\Omega, \mathcal{H}, \mathcal{H}_t, \theta_t, \xi_t, \Pi_x, \zeta)$ be an m -symmetric Borel right process on E . Here $\{\mathcal{H}_t : t \geq 0\}$ is the minimal admissible filtration, $\{\theta_t : t \geq 0\}$ the time-shift operator of ξ satisfying $\xi_t \circ \theta_s = \xi_{t+s}$ for $s, t \geq 0$, and $\zeta := \inf\{t > 0 : \xi_t = \partial\}$ the lifetime of ξ . Let $\{P_t : t \geq 0\}$ be the transition semigroup of ξ , i.e., for any non-negative measurable function f ,

$$P_t f(x) := \Pi_x [f(\xi_t)].$$

For $\alpha > 0$ and $f \in \mathcal{B}^+(E)$, let $G_\alpha f(x) := \int_0^{+\infty} e^{-\alpha t} P_t f(x) dt$. It is known by [5, Lemma 1.1.14] that $\{P_t : t \geq 0\}$ can be uniquely extended to a strongly continuous contraction semigroup on $L^2(E, m)$, which we also denote by $\{P_t : t \geq 0\}$. By the theory of Dirichlet forms, there exists a symmetric quasi-regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E, m)$ associated with ξ :

$$\mathcal{F} = \left\{ u \in L^2(E, m) : \sup_{t>0} \frac{1}{t} \int_E (u(x) - P_t u(x)) u(x) m(dx) < +\infty \right\},$$

$$\mathcal{E}(u, v) = \lim_{t \rightarrow 0} \frac{1}{t} \int_E (u(x) - P_t u(x)) v(x) m(dx), \quad \forall u, v \in \mathcal{F}.$$

Moreover, for all $f \in \mathcal{B}_b(E) \cap L^2(E, m)$ and $\alpha > 0$,

$$G_\alpha f \in \mathcal{F} \quad \text{with} \quad \mathcal{E}_\alpha(G_\alpha f, v) = (f, v) \quad \forall v \in \mathcal{F}, \quad (2.1)$$

where $\mathcal{E}_\alpha(u, v) := \mathcal{E}(u, v) + \alpha(u, v)$. It is known (cf. [21]) that this process is quasi-homeomorphic to a Hunt process associated with a regular Dirichlet form on a locally compact separable metric space. So all of the results of [21] apply to ξ and its Dirichlet form. Henceforth, we may and do assume ξ is an m -symmetric Hunt process on a locally compact separable metric space associated with a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$. We assume that ξ admits a transition density $p(t, x, y)$ with respect to the measure m , which is symmetric in (x, y) for each $t > 0$. Under this absolute continuity assumption, “quasi everywhere” statements can be strengthened to “everywhere” ones. Moreover, we can define notions without exceptional sets, for example, *positive continuous additive functionals in the strict sense* (cf. [21, Section 5.1]). In this paper, we will only deal with notions in the strict sense and omit “in the strict sense”. For convenience, for a measurable function f , we set

$$e_f(t) := \exp \left(- \int_0^t f(\xi_s) ds \right) \quad \forall t \geq 0,$$

whenever it is well defined.

In this paper, we consider a superprocess $X := \{X_t : t \geq 0\}$ with spatial motion ξ and a non-local branching mechanism ψ given by

$$\psi(x, f) = \phi^L(x, f(x)) + \phi^{NL}(x, f) \quad \text{for } x \in E, f \in \mathcal{B}_b^+(E). \quad (2.2)$$

The first term ϕ^L in (2.2) is called the local branching mechanism and takes the form

$$\phi^L(x, \lambda) = a(x)\lambda + b(x)\lambda^2 + \int_{(0, +\infty)} \left(e^{-\lambda\theta} - 1 + \lambda\theta \right) \Pi^L(x, d\theta) \quad \text{for } x \in E, \lambda \geq 0,$$

where $a(x) \in \mathcal{B}_b(E)$, $b(x) \in \mathcal{B}_b^+(E)$ and $(\theta \wedge \theta^2)\Pi^L(x, d\theta)$ is a bounded kernel from E to $(0, +\infty)$. The second term ϕ^{NL} in (2.2) is called the non-local branching mechanism and takes the form

$$\phi^{NL}(x, f) = -c(x)\pi(x, f) - \int_{(0, +\infty)} \left(1 - e^{-\theta\pi(x, f)} \right) \Pi^{NL}(x, d\theta) \quad \text{for } x \in E,$$

where $c(x)$ is a non-negative bounded measurable function on E , $\pi(x, dy)$ is a probability kernel on E with $\pi(x, \{x\}) \not\equiv 1$ and $\theta\Pi^{NL}(x, d\theta)$ is a bounded kernel from E to $(0, +\infty)$. To be specific, X is an $\mathcal{M}(E)$ -valued Markov process such that for every $f \in \mathcal{B}_b^+(E)$ and every $\mu \in \mathcal{M}(E)$,

$$P_\mu \left(e^{-\langle f, X_t \rangle} \right) = e^{-\langle u_f(\cdot, t), \mu \rangle} \quad \text{for } t \geq 0, \quad (2.3)$$

where $u_f(x, t) := -\log P_{\delta_x} \left(e^{-\langle f, X_t \rangle} \right)$ is the unique non-negative locally bounded solution to the integral equation

$$\begin{aligned} u_f(x, t) &= P_t f(x) - \Pi_x \left[\int_0^t \psi(\xi_s, u_f^{t-s}) ds \right] \\ &= P_t f(x) - \Pi_x \left[\int_0^t \phi^L(\xi_s, u_f(t-s, \xi_s)) ds \right] - \Pi_x \left[\int_0^t \phi^{NL}(\xi_s, u_f^{t-s}) ds \right]. \end{aligned} \quad (2.4)$$

We refer to the process described above as a (P_t, ϕ^L, ϕ^{NL}) -superprocess. It is known from [15] that the (P_t, ϕ^L, ϕ^{NL}) -superprocess can be constructed as the high density limit of some specific non-local branching particle systems, where whenever a particle dies at a point $x \in E$, it gives birth to a random number of offspring in E according to some probability kernels from E to the space of integer-valued finite measures on E , and the offspring then start to move from their locations of birth.

We define for $x \in E$,

$$\gamma(x) := c(x) + \int_{(0, +\infty)} \theta \Pi^{NL}(x, d\theta) \quad \text{and} \quad \gamma(x, dy) := \gamma(x)\pi(x, dy). \quad (2.5)$$

Clearly, $\gamma(x)$ is a non-negative bounded function on E , and $\gamma(x, dy)$ is a bounded kernel on E . Define

$$A := \{x \in E : \gamma(x) > 0\}.$$

Note that $\phi^{NL}(x, \cdot) = 0$ for all $x \in E \setminus A$. If $A = \emptyset$ (i.e., $\phi^{NL} \equiv 0$), we call ψ a (purely) local branching mechanism. Without loss of generality, we always assume that $A \neq \emptyset$. The arguments and results of this paper also work for (purely) local branching mechanisms.

It follows from [31, Theorem 5.12] that the (P_t, ϕ^L, ϕ^{NL}) -superprocess has a right realization in $\mathcal{M}(E)$. Let \mathcal{W}_0^+ denote the space of right continuous paths from $[0, +\infty)$ to $\mathcal{M}(E)$ having zero as a trap. Without loss of generality we assume X is the coordinate process in \mathcal{W}_0^+ and that $(\mathcal{F}_\infty, (\mathcal{F}_t)_{t \geq 0})$ is the natural filtration on \mathcal{W}_0^+ generated by the coordinate process. The following proposition follows from [31, Proposition 2.27 and Proposition 2.29].

Proposition 2.1. *For every $\mu \in \mathcal{M}(E)$ and $f \in \mathcal{B}_b(E)$,*

$$P_\mu(\langle f, X_t \rangle) = \langle \mathfrak{P}_t f, \mu \rangle,$$

where $\mathfrak{P}_t f(x)$ is the unique locally bounded solution to the following integral equation:

$$\mathfrak{P}_t f(x) = P_t f(x) - \Pi_x \left[\int_0^t a(\xi_s) \mathfrak{P}_{t-s} f(\xi_s) ds \right] + \Pi_x \left[\int_0^t \gamma(\xi_s, \mathfrak{P}_{t-s} f) ds \right]. \quad (2.6)$$

Moreover, for every $\mu \in \mathcal{M}(E)$, $g \in \mathcal{B}_b^+(E)$ and $f \in \mathcal{B}_b(E)$,

$$P_\mu \left(\langle f, X_t \rangle e^{-\langle g, X_t \rangle} \right) = e^{-\langle V_t g, \mu \rangle} \langle V_t^f g, \mu \rangle,$$

where $V_t g(x) := u_g(x, t)$ is the unique non-negative locally bounded solution to (2.4) with initial value g , and $V_t^f g(x)$ is the unique locally bounded solution to the following integral equation

$$V_t^f g(x) = P_t f(x) - \Pi_x \left[\int_0^t \Psi(\xi_s, V_{t-s} g, V_{t-s}^f g) ds \right], \quad (2.7)$$

where

$$\begin{aligned} \Psi(x, f, g) &:= g(x) \left(a(x) + 2b(x)f(x) + \int_{(0, +\infty)} \theta \left(1 - e^{-f(x)\theta} \right) \Pi^L(x, d\theta) \right) \\ &\quad - \pi(x, g) \left(c(x) + \int_{(0, +\infty)} \theta e^{-\theta \pi(x, f)} \Pi^{NL}(x, d\theta) \right). \end{aligned}$$

2.2 Kuznetsov measures

Let $\{Q_t(\mu, \cdot) := P_\mu(X_t \in \cdot) : t \geq 0, \mu \in \mathcal{M}(E)\}$ be the transition kernel of X . Then by (2.3), we have

$$\int_{\mathcal{M}(E)} e^{-\langle f, \nu \rangle} Q_t(\mu, d\nu) = \exp(-\langle V_t f, \mu \rangle) \quad \text{for } \mu \in \mathcal{M}(E) \text{ and } t \geq 0.$$

It implies that $Q_t(\mu_1 + \mu_2, \cdot) = Q_t(\mu_1, \cdot) * Q_t(\mu_2, \cdot)$ for any $\mu_1, \mu_2 \in \mathcal{M}(E)$, and hence $Q_t(\mu, \cdot)$ is an infinitely divisible probability measure on $\mathcal{M}(E)$. By the semigroup property of Q_t , V_t satisfies that

$$V_s V_t = V_{t+s} \quad \text{for all } s, t \geq 0.$$

Moreover, by the infinite divisibility of Q_t , each operator V_t has the representation

$$V_t f(x) = \lambda_t(x, f) + \int_{\mathcal{M}(E)^0} \left(1 - e^{-\langle f, \nu \rangle} \right) L_t(x, d\nu) \quad \text{for } t > 0, f \in \mathcal{B}_b^+(E), \quad (2.8)$$

where $\lambda_t(x, dy)$ is a bounded kernel on E and $(1 \wedge \nu(1))L_t(x, d\nu)$ is a bounded kernel from E to $\mathcal{M}(E)^0$. Let Q_t^0 be the restriction of Q_t to $\mathcal{M}(E)^0$. Let

$$E_0 := \{x \in E : \lambda_t(x, E) = 0 \text{ for all } t > 0\}.$$

If $x \in E_0$, then we get from (2.8) that

$$V_t f(x) = \int_{\mathcal{M}(E)^0} \left(1 - e^{-\langle f, \nu \rangle}\right) L_t(x, d\nu) \quad \text{for } t > 0, f \in \mathcal{B}_b^+(E).$$

It then follows from [31, Proposition 2.8 and Theorem A.40] that for every $x \in E_0$, the family of measures $\{L_t(x, \cdot) : t > 0\}$ on $\mathcal{M}(E)^0$ constitutes an entrance law for the restricted semigroup $\{Q_t^0 : t \geq 0\}$, and hence there corresponds a unique σ -finite measure \mathbb{N}_x on $(\mathcal{W}_0^+, \mathcal{F}_\infty)$ such that $\mathbb{N}_x(\{0\}) = 0$, and that for any $0 < t_1 < t_2 < \dots < t_n < +\infty$,

$$\mathbb{N}_x(X_{t_1} \in d\nu_1, X_{t_2} \in d\nu_2, \dots, X_{t_n} \in d\nu_n) = L_{t_1}(x, d\nu_1) Q_{t_2-t_1}^0(\nu_1, d\nu_2) \cdots Q_{t_n-t_{n-1}}^0(\nu_{n-1}, d\nu_n).$$

It immediately follows that for all $t > 0$ and $f \in \mathcal{B}_b^+(E)$,

$$\mathbb{N}_x \left(1 - e^{-\langle f, X_t \rangle}\right) = \int_{\mathcal{M}(E)^0} \left(1 - e^{-\langle f, \nu \rangle}\right) L_t(x, d\nu) = V_t f(x). \quad (2.9)$$

This measure \mathbb{N}_x is called the *Kuznetsov measure* corresponding to the entrance law $\{L_t(x, \cdot) : t > 0\}$ or the *excursion law* for the (P_t, ϕ^L, ϕ^{NL}) -superprocess. We refer to [31, section 8.4] for more details on the Kuznetsov measures. In the sequel, we assume that

Assumption 0. $E_+ := \{x \in E : b(x) > 0\} \subset E_0$.

Under this assumption, the Kuznetsov measure \mathbb{N}_x exists for every $x \in E_+$ when E_+ is nonempty. It is established in [12] that Assumption 0 is automatically true for superdiffusions with a (purely) local branching mechanism. [31, Theorem 8.6] also gives the following condition which is sufficient for Assumption 0: If there is a spatially independent local branching mechanism $\phi(\lambda)$ taking the form

$$\phi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0, +\infty)} \left(e^{-\lambda\theta} - 1 + \lambda\theta\right) n(d\theta) \quad \text{for } \lambda \geq 0,$$

where $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^+$ and $(\theta \wedge \theta^2)n(d\theta)$ is a bounded kernel on $(0, +\infty)$, such that $\phi'(\lambda) \rightarrow +\infty$ as $\lambda \rightarrow +\infty$, and that the branching mechanism ψ of X is bounded below by ϕ in the sense that

$$\psi(x, f) \geq \phi(f(x)) \quad \text{for all } x \in E \text{ and } f \in \mathcal{B}_b^+(E),$$

then $E_0 = E$.

3 Fundamental martingale

Definition 3.1. We call a non-negative measure μ on E a *smooth measure of ξ* if there is a positive continuous additive functional (PCAF in abbreviation) A_t^μ of ξ such that

$$\int_E f(x) \mu(dx) = \lim_{t \rightarrow 0} \frac{1}{t} \Pi_m \left[\int_0^t f(\xi_s) dA_s^\mu \right] \quad \text{for all } f \in \mathcal{B}^+(E).$$

Here $\Pi_m(\cdot) := \int_E \Pi_x(\cdot) m(dx)$. This measure μ is called the Revuz measure of A_t^μ . Moreover, we say that a smooth measure μ belongs to the Kato class $\mathbf{K}(\xi)$, if

$$\limsup_{t \downarrow 0} \int_0^t \int_E p(s, x, y) \mu(dy) ds = 0.$$

A function q is said to be in the class $\mathbf{K}(\xi)$ if $q(x)m(dx)$ is in $\mathbf{K}(\xi)$.

Clearly all bounded measurable functions are included in $\mathbf{K}(\xi)$. It is known (see, e.g., [1, Proposition 2.1.(i)] and [35, Theorem 3.1]) that if $\nu \in \mathbf{K}(\xi)$, then for every $\varepsilon > 0$ there is some constant $A_\varepsilon > 0$ such that

$$\int_E u(x)^2 \nu(dx) \leq \varepsilon \mathcal{E}(u, u) + A_\varepsilon \int_E u(x)^2 m(dx) \quad \forall u \in \mathcal{F}. \quad (3.1)$$

Assumption 1. $\int_E \pi(x, \cdot) m(dx) \in \mathbf{K}(\xi)$.

Under Assumption 1, it follows from (3.1), the boundedness of $\gamma(x)$ and the inequality

$$|u(x)u(y)| \leq \frac{1}{2}(u(x)^2 + u(y)^2)$$

that for every $\varepsilon > 0$, there is a constant $K_\varepsilon > 0$ such that

$$\int_E \int_E u(x)u(y)\gamma(x, dy)m(dx) \leq \varepsilon \mathcal{E}(u, u) + K_\varepsilon \int_E u(x)^2 m(dx) \quad \forall u \in \mathcal{F}.$$

It follows that the bilinear form $(\mathcal{Q}, \mathcal{F})$ defined by

$$\mathcal{Q}(u, v) := \mathcal{E}(u, v) + \int_E a(x)u(x)v(x)m(dx) - \int_E \int_E u(y)v(x)\gamma(x, dy)m(dx) \quad \forall u, v \in \mathcal{F},$$

is closed and that there are positive constants K and β_0 such that $\mathcal{Q}_{\beta_0}(u, u) := \mathcal{Q}(u, u) + \beta_0(u, u) \geq 0$ for all $u \in \mathcal{F}$, and

$$|\mathcal{Q}(u, v)| \leq K \mathcal{Q}_{\beta_0}(u, u)^{1/2} \mathcal{Q}_{\beta_0}(v, v)^{1/2} \quad \forall u, v \in \mathcal{F}.$$

It then follows from [26] that for such a closed form $(\mathcal{Q}, \mathcal{F})$ on $L^2(E, m)$, there are unique strongly continuous semigroups $\{T_t : t \geq 0\}$ and $\{\widehat{T}_t : t \geq 0\}$ on $L^2(E, m)$ such that $\|T_t\|_{L^2(E, m)} \leq e^{\beta_0 t}$, $\|\widehat{T}_t\|_{L^2(E, m)} \leq e^{\beta_0 t}$, and

$$(T_t f, g) = (f, \widehat{T}_t g) \quad \forall f, g \in L^2(E, m). \quad (3.2)$$

Let $\{U_\alpha\}_{\alpha > \beta_0}$ and $\{\widehat{U}_\alpha\}_{\alpha > \beta_0}$ be given by $U_\alpha f := \int_0^{+\infty} e^{-\alpha t} T_t f dt$ and $\widehat{U}_\alpha f := \int_0^{+\infty} e^{-\alpha t} \widehat{T}_t f dt$ respectively. Then $\{U_\alpha\}_{\alpha > \beta_0}$ and $\{\widehat{U}_\alpha\}_{\alpha > \beta_0}$ are strongly continuous pseudo-resolvents in the sense that they satisfy the resolvent equations

$$U_\alpha - U_\beta + (\alpha - \beta)U_\alpha U_\beta = 0, \quad \widehat{U}_\alpha - \widehat{U}_\beta + (\alpha - \beta)\widehat{U}_\alpha \widehat{U}_\beta = 0$$

for all $\alpha, \beta > \beta_0$, and

$$\mathcal{Q}_\alpha(U_\alpha f, g) = \mathcal{Q}_\alpha(g, \widehat{U}_\alpha f) = (f, g) \quad \forall f \in L^2(E, m), g \in \mathcal{F}. \quad (3.3)$$

Recall that \mathfrak{P}_t is the mean semigroup of the (P_t, Φ^L, Φ^{NL}) -superprocess, which satisfies that for every $f \in \mathcal{B}_b(E)$,

$$\mathfrak{P}_t f(x) = P_t f(x) - \Pi_x \left[\int_0^t a(\xi_s) \mathfrak{P}_{t-s} f(\xi_s) ds \right] + \Pi_x \left[\int_0^t \gamma(\xi_s, \mathfrak{P}_{t-s} f) ds \right]. \quad (3.4)$$

Since $\gamma(x, dy) = \gamma(x) \pi(x, dy)$ and $a(x)$, $\gamma(x)$ are bounded functions on E , by (3.4), we have for every $f \in \mathcal{B}_b(E)$,

$$\|\mathfrak{P}_t f\|_\infty \leq \|f\|_\infty + (\|a\|_\infty + \|\gamma\|_\infty) \int_0^t \|\mathfrak{P}_{t-s} f\|_\infty ds.$$

By Gronwall's lemma, $\|\mathfrak{P}_t f\|_\infty \leq e^{c_1 t} \|f\|_\infty$ for some constant $c_1 > 0$. For $f \in \mathcal{B}_b(E)$ and $\alpha > c_1$, define $R_\alpha f(x) := \int_0^{+\infty} e^{-\alpha t} \mathfrak{P}_t f(x) dt$. By taking Laplace transform of both sides of (3.4), we have

$$R_\alpha f(x) = G_\alpha f(x) - G_\alpha (\alpha R_\alpha f)(x) + G_\alpha (\gamma(\cdot, R_\alpha f))(x), \quad (3.5)$$

where G_α is the α -resolvent of $(P_t)_{t \geq 0}$. A particular case is when $a(x), \gamma(x) \in L^2(E, m)$. In this case, for all $f \in \mathcal{B}_b(E) \cap L^2(E, m)$ and α sufficiently large, both $a(x)R_\alpha f(x)$ and $\gamma(x, R_\alpha f)$ are in $\mathcal{B}_b(E) \cap L^2(E, m)$. Then it follows from (2.1) that $G_\alpha f, G_\alpha (\alpha R_\alpha f), G_\alpha (\gamma(\cdot, R_\alpha f)) \in \mathcal{F}$, and then by (3.5), (2.1) and (3.3),

$$\mathcal{Q}_\alpha(R_\alpha f, v) = (f, v) = \mathcal{Q}_\alpha(U_\alpha f, v) \quad \text{for all } v \in \mathcal{F},$$

which implies that $R_\alpha f$ is m -equivalent to $U_\alpha f$ for α sufficiently large. This indicates that there is some strong relation between \mathfrak{P}_t and T_t . In fact we will show in Proposition 5.2 below that $\mathfrak{P}_t f = T_t f [m]$ for every $t > 0$ and every $f \in \mathcal{B}_b(E) \cap L^2(E, m)$. This means that \mathfrak{P}_t can be regarded as a bounded linear operator on the space of bounded measurable functions in $L^2(E, m)$, which is dense in $L^2(E, m)$. Hence T_t can be regarded as the unique bounded linear operator on $L^2(E, m)$ which is an extension of \mathfrak{P}_t .

Assumption 2. There exist a constant $\lambda_1 \in (-\infty, +\infty)$ and positive functions $h, \hat{h} \in \mathcal{F}$ with h bounded continuous, $\|h\|_{L^2(E, m)} = 1$ and $(h, \hat{h}) = 1$ such that

$$\mathcal{Q}(h, v) = \lambda_1(h, v), \quad \mathcal{Q}(v, \hat{h}) = \lambda_1(v, \hat{h}) \quad \forall v \in \mathcal{F}. \quad (3.6)$$

This equation implies that $T_t h = e^{-\lambda_1 t} h$ and $\hat{T}_t \hat{h} = e^{-\lambda_1 t} \hat{h}$ in $L^2(E, m)$.

Since $h \in \mathcal{F}$ is continuous, by Fukushima's decomposition, we have for every $x \in E$, Π_x -a.s.

$$h(\xi_t) - h(\xi_0) = M_t^h + N_t^h, \quad t \geq 0,$$

where M^h is a martingale additive functional of ξ having finite energy and N_t^h is a continuous additive functional of ξ having zero energy. It follows from (3.6) and [21, Theorem 5.4.2] that N_t^h is of bounded variation, and

$$N_t^h = -\lambda_1 \int_0^t h(\xi_s) ds + \int_0^t a(\xi_s) h(\xi_s) ds - \int_0^t \gamma(\xi_s, h) ds, \quad \forall t \geq 0.$$

Following the idea of [6, Section 2], we define a local martingale on the random time interval $[0, \zeta_p)$ by

$$M_t := \int_0^t \frac{1}{h(\xi_{s-})} dM_s^h, \quad t \in [0, \zeta_p), \quad (3.7)$$

where ζ_p is the predictable part of the lifetime ζ of ξ . Then the solution H_t of the stochastic differential equation

$$H_t = 1 + \int_0^t H_{s-} dM_s, \quad t \in [0, \zeta_p), \quad (3.8)$$

is a positive local martingale on $[0, \zeta_p)$ and hence a supermartingale. Consequently, the formula

$$d\Pi_x^h = H_t d\Pi_x \quad \text{on } \mathcal{H}_t \cap \{t < \zeta\} \quad \text{for } x \in E$$

uniquely determines a family of subprobability measures $\{\Pi_x^h : x \in E\}$ on (Ω, \mathcal{H}) . Hence we have

$$\Pi_x^h[f(\xi_t)] = \Pi_x[H_t f(\xi_t); t < \zeta]$$

for any $t \geq 0$ and $f \in \mathcal{B}^+(E)$. Note that by (3.7), (3.8) and Doléan-Dade's formula,

$$H_t = \exp\left(M_t - \frac{1}{2}\langle M^c \rangle_t\right) \prod_{0 \leq s \leq t} \frac{h(\xi_s)}{h(\xi_{s-})} \exp\left(1 - \frac{h(\xi_s)}{h(\xi_{s-})}\right), \quad t \in [0, \zeta_p), \quad (3.9)$$

where M^c is the continuous martingale part of M . Applying Ito's formula to $\log h(\xi_t)$, we obtain that for every $x \in E$, Π_x -a.s. on $[0, \zeta)$,

$$\begin{aligned} \log h(\xi_t) - \log h(\xi_0) &= M_t - \frac{1}{2}\langle M^c \rangle_t + \sum_{s \leq t} \left(\log \frac{h(\xi_s)}{h(\xi_{s-})} - \frac{h(\xi_s) - h(\xi_{s-})}{h(\xi_{s-})} \right) \\ &\quad - \lambda_1 t + \int_0^t a(\xi_s) ds - \int_0^t \frac{\gamma(\xi_s, h)}{h(\xi_s)} ds. \end{aligned} \quad (3.10)$$

Put

$$q(x) := \gamma(x, h)/h(x) \quad \text{for } x \in E. \quad (3.11)$$

By (3.9) and (3.10), we get

$$H_t = \exp\left(\lambda_1 t - \int_0^t a(\xi_s) ds + \int_0^t q(\xi_s) ds\right) \frac{h(\xi_t)}{h(\xi_0)}.$$

To emphasize, the process ξ under $\{\Pi_x^h, x \in E\}$ will be denoted as ξ^h . Therefore for any $f \in \mathcal{B}^+(E)$ and $t \geq 0$,

$$P_t^h f(x) := \Pi_x^h[f(\xi_t^h)] = \frac{e^{\lambda_1 t}}{h(x)} \Pi_x[e_{a-q}(t) h(\xi_t) f(\xi_t)]. \quad (3.12)$$

It follows from [6, Theorem 2.6] that the transformed process ξ^h is a conservative and recurrent (in the sense of [21]) \tilde{m} -symmetric right Markov process on E with $\tilde{m}(dy) := h(y)^2 m(dy)$. Thus

$$P_t^h 1 = 1 \quad [\tilde{m}] \quad \text{for all } t > 0. \quad (3.13)$$

Note that for every $t > 0$ and $x \in E$, the measure $P_t^h(x, \cdot) := \Pi_x^h(\xi_t^h \in \cdot)$ is absolutely continuous with respect to \tilde{m} , since $P_t^h(x, \cdot)$ is absolutely continuous with respect to the measure $P_t(x, \cdot) := \Pi_x(\xi_t \in \cdot)$ by (3.12) and the latter is absolutely continuous with respect to m . Moreover, by the right continuity of the sample paths of ξ^h , one can easily verify that both 1 and $P_t^h 1(x)$ are excessive functions for $\{P_t^h : t > 0\}$. Thus by [5, Theorem A.2.17], (3.13) implies that

$$1 = P_t^h 1(x) = \frac{e^{\lambda_1 t}}{h(x)} \Pi_x [e_{a-q}(t) h(\xi_t)] \quad \text{for all } x \in E. \quad (3.14)$$

Theorem 3.2. *For every $\mu \in \mathcal{M}(E)$, $W_t^h(X) := e^{\lambda_1 t} \langle h, X_t \rangle$ is a non-negative P_μ -martingale with respect to the filtration $\{\mathcal{F}_t : t \geq 0\}$.*

Proof. By the Markov property of X , it suffices to prove that for any $x \in E$ and $t \geq 0$,

$$\mathfrak{P}_t h(x) = P_{\delta_x}(\langle h, X_t \rangle) = e^{-\lambda_1 t} h(x). \quad (3.15)$$

Let $A(s, t) := -\int_s^t a(\xi_r) dr + \int_s^t q(\xi_r) dr$ and $u(t, x) := \Pi_x [e^{A(0, t)} h(\xi_t)]$. Clearly by (3.14), $u(t, x) = e^{-\lambda_1 t} h(x)$. Note that

$$e^{A(0, t)} - 1 = -(e^{A(t, t)} - e^{A(0, t)}) = \int_0^t (-a(\xi_s) + q(\xi_s)) e^{A(s, t)} ds. \quad (3.16)$$

By (3.16), Fubini's theorem and the Markov property of ξ ,

$$\begin{aligned} u(t, x) &= P_t h(x) + \Pi_x [(e^{A(0, t)} - 1) h(\xi_t)] \\ &= P_t h(x) - \Pi_x \left[\int_0^t a(\xi_s) e^{A(s, t)} h(\xi_t) ds \right] + \Pi_x \left[\int_0^t q(\xi_s) e^{A(s, t)} h(\xi_t) ds \right] \\ &= P_t h(x) - \Pi_x \left[\int_0^t a(\xi_s) u(t-s, \xi_s) ds \right] + \Pi_x \left[\int_0^t \frac{\gamma(\xi_s, h)}{h(\xi_s)} u(t-s, \xi_s) ds \right] \\ &= P_t h(x) - \Pi_x \left[\int_0^t a(\xi_s) u(t-s, \xi_s) ds \right] + \Pi_x \left[\int_0^t \gamma(\xi_s, u^{t-s}) ds \right]. \end{aligned}$$

In the last equality above we used the fact that $u(t-s, x) = e^{-\lambda_1(t-s)} h(x)$. Thus $u(t, x)$ is a locally bounded solution to (2.6) with initial value h . Hence by the uniqueness of the solution, we get $u(t, x) = \mathfrak{P}_t h(x) = P_{\delta_x}(\langle h, X_t \rangle)$. \square

For $\mu \in \mathcal{M}(E)$, we say that the process X exhibits *weak local extinction* under P_μ if for every nonempty relatively compact open subset B of E , $P_\mu(\lim_{t \rightarrow +\infty} X_t(B) = 0) = 1$.

Corollary 3.3. *For every $\mu \in \mathcal{M}(E)$ and every nonempty relatively compact open subset B of E ,*

$$P_\mu \left(\limsup_{t \rightarrow +\infty} e^{\lambda_1 t} X_t(B) < +\infty \right) = 1.$$

In particular, if $\lambda_1 > 0$, then X exhibits weak local extinction under P_μ .

Proof. This corollary follows immediately from Theorem 3.2 and the fact that

$$e^{\lambda_1 t} X_t(B) \leq e^{\lambda_1 t} \left\langle \frac{h}{\inf_{x \in B} h(x)} 1_B, X_t \right\rangle \leq \frac{1}{\inf_{x \in B} h(x)} W_t^h(X).$$

\square

Remark 3.4. Corollary 3.3 implies that the local mass of X_t grows subexponentially and the growth rate can not exceed $-\lambda_1$. However when one considers the total mass process $\langle 1, X_t \rangle$, the growth rate may actually exceed $-\lambda_1$. We refer to [17] and [18] for more concrete examples.

4 Spine decomposition

4.1 Concatenation process

It is well-known (see, e.g., [34, p. 286]) that for every $x \in E$, there is a unique (up to equivalence in law) right process $((\hat{\xi}_t)_{t \geq 0}; \hat{\Pi}_x^h)$ on E with lifetime $\hat{\zeta}$ and terminal point ∂ , such that

$$\hat{\Pi}_x^h(\hat{\xi}_t \in B) = \Pi_x^h[e_q(t); \xi_t^h \in B] \quad \forall B \in \mathcal{B}(E).$$

$\hat{\xi}$ is called the $e_q(t)$ -subprocess of ξ^h . In fact, a version of the $e_q(t)$ -subprocess can be obtained by the following method of curtailment of the lifetime. Let Z be an exponential random variable, of parameter 1, independent of ξ^h . Put

$$\hat{\zeta}(\omega) := \inf\{t \geq 0 : \int_0^t q(\xi_s^h(\omega)) ds \geq Z(\omega)\} (= +\infty, \text{ if such } t \text{ does not exist}),$$

and

$$\hat{\xi}_t(\omega) := \begin{cases} \xi_t^h(\omega) & \text{if } t < \hat{\zeta}(\omega), \\ \partial & \text{if } t \geq \hat{\zeta}(\omega). \end{cases}$$

Then the process $((\hat{\xi}_t)_{t \geq 0}, \Pi_x^h)$ is equal in law to the $e_q(t)$ -subprocess of ξ^h . Now we define

$$\pi^h(x, dy) := \frac{h(y)\pi(x, dy)}{\pi(x, h)} \quad \text{for } x \in E. \quad (4.1)$$

Obviously, $\pi^h(x, dy)$ is a probability kernel on E . Let $\tilde{\xi} := (\tilde{\Omega}, \tilde{\mathcal{G}}, \tilde{\mathcal{G}}_t, \tilde{\theta}_t, \tilde{\xi}_t, \tilde{\Pi}_x, \tilde{\zeta})$ be the right process constructed from $\hat{\xi}$ and the instantaneous distribution $\kappa(\omega, dy) := \pi^h(\hat{\xi}_{\hat{\zeta}(\omega)-}(\omega), dy)$ by using the so-called “piecing out” procedure (cf. Ikeda et al. [25]). We will follow the terminology of [34, Section II.14] and call $\tilde{\xi}$ a *concatenation* process defined from an infinite sequence of copies of $\hat{\xi}$ and the transfer kernel $\kappa(\omega, dy)$. One can also refer to [31, Section A.6] for a summary of concatenation processes. The intuitive idea of this concatenation is described as follows. The process $\tilde{\xi}$ evolves as the process ξ^h until time $\hat{\zeta}$, it is then revived by means of the kernel $\kappa(\omega, dy)$ and evolves again as ξ^h and so on, until a countably infinite number of revivals have occurred. Clearly in the case of purely local branching mechanism (i.e. $\gamma(x) \equiv 0$ on E), we have $\hat{\zeta} = +\infty$ almost surely and hence $\tilde{\xi}$ runs as a copy of ξ^h .

Let \tilde{P}_t be the transition semigroup of $\tilde{\xi}$. We have the renewal equation

$$\tilde{P}_t f(x) = \Pi_x^h[e_q(t)f(\xi_t^h)] + \Pi_x^h\left[\int_0^t q(\xi_s^h)e_q(s)\pi^h(\xi_s^h, \tilde{P}_{t-s}f)ds\right] \quad (4.2)$$

for every $f \in \mathcal{B}_b^+(E)$, see for instance, [31, Section A.6]. By [31, Proposition 2.9], the above equation can be rewritten as

$$\tilde{P}_t f(x) = \Pi_x^h[f(\xi_t^h)] - \Pi_x^h\left[\int_0^t q(\xi_s^h)\tilde{P}_{t-s}f(\xi_s^h)ds\right] + \Pi_x^h\left[\int_0^t q(\xi_s^h)\pi^h(\xi_s^h, \tilde{P}_{t-s}f)ds\right].$$

Proposition 4.1. *For every $f \in \mathcal{B}_b^+(E)$, $t \geq 0$ and $x \in E$,*

$$\tilde{P}_t f(x) = \frac{e^{\lambda_1 t}}{h(x)} \mathfrak{P}_t(fh)(x). \quad (4.3)$$

In particular $\tilde{P}_t 1(x) \equiv 1$, hence $\tilde{\xi}$ has infinite lifetime. Moreover, for each $t > 0$ and $x \in E$, $\tilde{\xi}$ has a transition density $\tilde{p}(t, x, y)$ with respect to the probability measure

$$\rho(dy) := h(y) \hat{h}(y) m(dy). \quad (4.4)$$

Proof. By (4.2), (3.12), (2.5), (3.11) and (4.1), we have

$$\begin{aligned} \tilde{P}_t f(x) &= \frac{e^{\lambda_1 t}}{h(x)} \Pi_x [e_a(t) h(\xi_t) f(\xi_t)] + \frac{e^{\lambda_1 t}}{h(x)} \Pi_x \left[\int_0^t e_a(s) q(\xi_s) h(\xi_s) e^{-\lambda_1(t-s)} \pi^h(\xi_s, \tilde{P}_{t-s} f) ds \right] \\ &= \frac{e^{\lambda_1 t}}{h(x)} \Pi_x [e_a(t) h(\xi_t) f(\xi_t)] + \frac{e^{\lambda_1 t}}{h(x)} \Pi_x \left[\int_0^t e_a(s) \gamma(\xi_s, e^{-\lambda_1(t-s)} h \tilde{P}_{t-s} f) ds \right]. \end{aligned} \quad (4.5)$$

Let $u(t, x) := e^{-\lambda_1 t} h(x) \tilde{P}_t f(x)$. Clearly $u(t, x)$ is a locally bounded function on $[0, +\infty) \times E$. Moreover it follows from (4.5) and [31, Proposition 2.9] that

$$\begin{aligned} u(t, x) &= \Pi_x [e_a(t) h(\xi_t) f(\xi_t)] + \Pi_x \left[\int_0^t e_a(s) \gamma(\xi_s, u^{t-s}) ds \right] \\ &= \Pi_x [h(\xi_t) f(\xi_t)] - \Pi_x \left[\int_0^t a(\xi_s) u(t-s, \xi_s) ds \right] + \Pi_x \left[\int_0^t \gamma(\xi_s, u^{t-s}) ds \right]. \end{aligned}$$

This implies that $u(t, x)$ is a locally bounded solution to (2.6) with initial value fh . Hence we get $e^{-\lambda_1 t} h(x) \tilde{P}_t f(x) = u(t, x) = \mathfrak{P}_t(fh)(x)$ by the uniqueness of the solution. It then follows from (3.15) that $\tilde{P}_t 1(x) \equiv 1$ on E .

To prove the second part of this proposition, it suffices to prove that for each $t > 0$ and $x \in E$, $\tilde{P}_t 1_B(x) = 0$ for all $B \in \mathcal{B}(E)$ with $\rho(B) = 0$ (or equivalently, $m(B) = 0$). Note that $\Pi_x [h 1_B(\xi_t)] = \int_B p(t, x, y) m(dy) = 0$. It follows from the above argument that $e^{-\lambda_1 t} h(x) \tilde{P}_t 1_B(x) = \mathfrak{P}_t(h 1_B)(x)$ is the unique locally bounded solution to (2.6) with initial value 0. Thus $\tilde{P}_t 1_B(x) \equiv 0$. \square

Remark 4.2. The formula (4.3) can be written as

$$\frac{P_{\delta_x} [\langle fh, X_t \rangle]}{P_{\delta_x} [\langle h, X_t \rangle]} = \tilde{\Pi}_x [f(\tilde{\xi}_t)] \quad \text{for } f \in \mathcal{B}_b^+(E) \text{ and } t \geq 0, \quad (4.6)$$

which enables us to calculate the first moment of the superprocess in terms of an auxiliary process. An analogous formula for a special class of non-local branching Markov processes, which is called a “many-to-one” formula, is established in [2], but with a totally different method. By (3.15), we may rewrite (4.6) as

$$P_{\delta_x} [\langle fh, X_t \rangle] = e^{\lambda_1 t} h(x) \tilde{\Pi}_x [f(\tilde{\xi}_t)] \quad \text{for } f \in \mathcal{B}_b^+(E) \text{ and } t \geq 0.$$

Let τ_1 be the first revival time of $\tilde{\xi}$. For $n \geq 2$, define τ_n recursively by $\tau_n := \tau_{n-1} + \tau_1 \circ \tilde{\theta}_{\tau_{n-1}}$. Since $\tilde{\xi}$ has infinite lifetime, $\tilde{\Pi}_x (\lim_{n \rightarrow +\infty} \tau_n = +\infty) = 1$ for all $x \in E$.

Proposition 4.3. For every $f(s, x, y), g(s, x, y) \in \mathcal{B}^+([0, +\infty) \times E \times E)$, $t > 0$ and $x \in E$, we have

$$\tilde{\Pi}_x \left[\sum_{\tau_i \leq t} f(\tau_i, \tilde{\xi}_{\tau_i-}, \tilde{\xi}_{\tau_i}) \right] = \tilde{\Pi}_x \left[\int_0^t q(\tilde{\xi}_s) ds \int_E f(s, \tilde{\xi}_s, y) \pi^h(\tilde{\xi}_s, dy) \right] \quad (4.7)$$

and

$$\begin{aligned} & \tilde{\Pi}_x \left[\left(\sum_{\tau_i \leq t} f(\tau_i, \tilde{\xi}_{\tau_i-}, \tilde{\xi}_{\tau_i}) \right) \left(\sum_{\tau_j \leq t} g(\tau_j, \tilde{\xi}_{\tau_j-}, \tilde{\xi}_{\tau_j}) \right) \right] \\ &= \tilde{\Pi}_x \left[\sum_{\tau_i \leq t} f g(\tau_i, \tilde{\xi}_{\tau_i-}, \tilde{\xi}_{\tau_i}) \right] \\ &+ \tilde{\Pi}_x \left[\int_0^t q(\tilde{\xi}_s) ds \int_E f(s, \tilde{\xi}_s, y) \tilde{\Pi}_y \left(\int_0^{t-s} q(\tilde{\xi}_r) dr \int_E g(s+r, \tilde{\xi}_r, z) \pi^h(\tilde{\xi}_r, dz) \right) \pi^h(\tilde{\xi}_s, dy) \right] \\ &+ \tilde{\Pi}_x \left[\int_0^t q(\tilde{\xi}_s) ds \int_E g(s, \tilde{\xi}_s, y) \tilde{\Pi}_y \left(\int_0^{t-s} q(\tilde{\xi}_r) dr \int_E f(s+r, \tilde{\xi}_r, z) \pi^h(\tilde{\xi}_r, dz) \right) \pi^h(\tilde{\xi}_s, dy) \right]. \end{aligned} \quad (4.8)$$

Proof. We will prove (4.7) first. We claim that

$$\tilde{\Pi}_x \left[f(\tau_1, \tilde{\xi}_{\tau_1-}, \tilde{\xi}_{\tau_1}) 1_{\{\tau_1 \leq t\}} \right] = \tilde{\Pi}_x \left[\int_0^{t \wedge \tau_1} q(\tilde{\xi}_s) ds \int_E f(s, \tilde{\xi}_s, y) \pi^h(\tilde{\xi}_s, dy) \right]. \quad (4.9)$$

It is easy to see from the construction of $\tilde{\xi}$ that

$$\text{LHS of (4.9)} = \Pi_x^h \left[\int_0^t q(\xi_s^h) e_q(s) ds \int_E f(s, \xi_s^h, y) \pi^h(\xi_s^h, dy) \right]. \quad (4.10)$$

On the other hand, by Fubini's theorem, we have

$$\begin{aligned} \text{RHS of (4.9)} &= \int_0^t ds \tilde{\Pi}_x \left[\int_E q(\tilde{\xi}_s) f(s, \tilde{\xi}_s, y) \pi^h(\tilde{\xi}_s, dy) 1_{\{s < \tau_1\}} \right] \\ &= \int_0^t ds \Pi_x^h \left[\int_E q(\hat{\xi}_s) f(s, \hat{\xi}_s, y) \pi^h(\hat{\xi}_s, dy) 1_{\{s < \hat{\zeta}\}} \right] \\ &= \int_0^t ds \Pi_x^h \left[e_q(s) \int_E q(\xi_s^h) f(s, \xi_s^h, y) \pi^h(\xi_s^h, dy) \right] \\ &= \Pi_x^h \left[\int_0^t q(\xi_s^h) e_q(s) ds \int_E f(s, \xi_s^h, y) \pi^h(\xi_s^h, dy) \right]. \end{aligned} \quad (4.11)$$

Combining (4.10) and (4.11) we arrive at the claim (4.9). Note that applying the shift operator $\tilde{\theta}_{\tau_n}$ to $f(\tau_1, \tilde{\xi}_{\tau_1-}, \tilde{\xi}_{\tau_1}) 1_{\{\tau_1 \leq t\}}$ gives $f(\tau_{n+1}, \tilde{\xi}_{\tau_{n+1}-}, \tilde{\xi}_{\tau_{n+1}}) 1_{\{\tau_{n+1} \leq t\}}$. Using the strong Markov property of $\tilde{\xi}$ and Fubini's theorem, we can prove by induction that for all $n \geq 2$,

$$\tilde{\Pi}_x \left[f(\tau_n, \tilde{\xi}_{\tau_n-}, \tilde{\xi}_{\tau_n}) 1_{\{\tau_n \leq t\}} \right] = \tilde{\Pi}_x \left[\int_{t \wedge \tau_{n-1}}^{t \wedge \tau_n} q(\tilde{\xi}_s) ds \int_E f(s, \tilde{\xi}_s, y) \pi^h(\tilde{\xi}_s, dy) \right]. \quad (4.12)$$

Thus by (4.12), Fubini's theorem and the fact that $\tilde{\Pi}_x(\lim_{n \rightarrow +\infty} \tau_n = +\infty) = 1$, we have

$$\begin{aligned} \tilde{\Pi}_x \left[\sum_{\tau_i \leq t} f(\tau_i, \tilde{\xi}_{\tau_i-}, \tilde{\xi}_{\tau_i}) \right] &= \tilde{\Pi}_x \left[\sum_{i=1}^{+\infty} f(\tau_i, \tilde{\xi}_{\tau_i-}, \tilde{\xi}_{\tau_i}) 1_{\{\tau_i \leq t\}} \right] \\ &= \tilde{\Pi}_x \left[\lim_{n \rightarrow +\infty} \int_0^{t \wedge \tau_n} q(\tilde{\xi}_s) ds \int_E f(s, \tilde{\xi}_s, y) \pi^h(\tilde{\xi}_s, dy) \right] \\ &= \tilde{\Pi}_x \left[\int_0^t q(\tilde{\xi}_s) ds \int_E f(s, \tilde{\xi}_s, y) \pi^h(\tilde{\xi}_s, dy) \right]. \end{aligned}$$

Hence we have proved (4.7). We next show (4.8). It is easy to see that

$$\begin{aligned} &\tilde{\Pi}_x \left[\left(\sum_{\tau_i \leq t} f(\tau_i, \tilde{\xi}_{\tau_i-}, \tilde{\xi}_{\tau_i}) \right) \left(\sum_{\tau_j \leq t} g(\tau_j, \tilde{\xi}_{\tau_j-}, \tilde{\xi}_{\tau_j}) \right) \right] \\ &= \tilde{\Pi}_x \left[\sum_{\tau_i \leq t} f g(\tau_i, \tilde{\xi}_{\tau_i-}, \tilde{\xi}_{\tau_i}) \right] + \sum_{i=1}^{+\infty} \sum_{j=i+1}^{+\infty} \left\{ \tilde{\Pi}_x \left[f(\tau_i, \tilde{\xi}_{\tau_i-}, \tilde{\xi}_{\tau_i}) g(\tau_j, \tilde{\xi}_{\tau_j-}, \tilde{\xi}_{\tau_j}) 1_{\{\tau_j \leq t\}} \right] \right. \\ &\quad \left. + \tilde{\Pi}_x \left[g(\tau_i, \tilde{\xi}_{\tau_i-}, \tilde{\xi}_{\tau_i}) f(\tau_j, \tilde{\xi}_{\tau_j-}, \tilde{\xi}_{\tau_j}) 1_{\{\tau_j \leq t\}} \right] \right\}. \end{aligned} \quad (4.13)$$

By the strong Markov property and (4.10), we have for $j \geq 2$,

$$\begin{aligned} &\tilde{\Pi}_x \left[f(\tau_1, \tilde{\xi}_{\tau_1-}, \tilde{\xi}_{\tau_1}) g(\tau_j, \tilde{\xi}_{\tau_j-}, \tilde{\xi}_{\tau_j}) 1_{\{\tau_j \leq t\}} \right] \\ &= \tilde{\Pi}_x \left[f(\tau_1, \tilde{\xi}_{\tau_1-}, \tilde{\xi}_{\tau_1}) 1_{\{\tau_1 \leq t\}} \tilde{\Pi}_{\tilde{\xi}_{\tau_1}} \left(g(\tau_{j-1} + s, \tilde{\xi}_{\tau_{j-1}-}, \tilde{\xi}_{\tau_{j-1}}) 1_{\{\tau_{j-1} + s \leq t\}} \right) \Big|_{s=\tau_1} \right] \\ &= \Pi_x^h \left[\int_0^t q(\xi_s^h) e_q(s) ds \int_E f(s, \xi_s^h, y) \tilde{\Pi}_y \left(g(\tau_{j-1} + s, \tilde{\xi}_{\tau_{j-1}-}, \tilde{\xi}_{\tau_{j-1}}) 1_{\{\tau_{j-1} \leq t-s\}} \right) \pi^h(\xi_s^h, dy) \right], \end{aligned} \quad (4.14)$$

and for $j > i \geq 2$,

$$\begin{aligned} &\tilde{\Pi}_x \left[f(\tau_i, \tilde{\xi}_{\tau_i-}, \tilde{\xi}_{\tau_i}) g(\tau_j, \tilde{\xi}_{\tau_j-}, \tilde{\xi}_{\tau_j}) 1_{\{\tau_j \leq t\}} \right] \\ &= \tilde{\Pi}_x \left[1_{\{\tau_i \leq t\}} \tilde{\Pi}_{\tilde{\xi}_{\tau_i}} \left(f(\tau_{i-1} + s, \tilde{\xi}_{\tau_{i-1}-}, \tilde{\xi}_{\tau_{i-1}}) g(\tau_{j-1} + s, \tilde{\xi}_{\tau_{j-1}-}, \tilde{\xi}_{\tau_{j-1}}) 1_{\{\tau_{j-1} + s \leq t\}} \right) \Big|_{s=\tau_i} \right] \\ &= \Pi_x^h \left[\int_0^t q(\xi_s^h) e_q(s) ds \int_E \tilde{\Pi}_y \left(f(\tau_{i-1} + s, \tilde{\xi}_{\tau_{i-1}-}, \tilde{\xi}_{\tau_{i-1}}) g(\tau_{j-1} + s, \tilde{\xi}_{\tau_{j-1}-}, \tilde{\xi}_{\tau_{j-1}}) 1_{\{\tau_{j-1} \leq t-s\}} \right) \pi^h(\xi_s^h, dy) \right]. \end{aligned} \quad (4.15)$$

By (4.14), Fubini's theorem, the strong Markov property of $\tilde{\xi}$, (4.7) and (4.10),

$$\begin{aligned} &\sum_{j=2}^{+\infty} \tilde{\Pi}_x \left[f(\tau_1, \tilde{\xi}_{\tau_1-}, \tilde{\xi}_{\tau_1}) g(\tau_j, \tilde{\xi}_{\tau_j-}, \tilde{\xi}_{\tau_j}) 1_{\{\tau_j \leq t\}} \right] \\ &= \Pi_x^h \left[\int_0^t q(\xi_s^h) e_q(s) ds \int_E f(s, \xi_s^h, y) \tilde{\Pi}_y \left(\sum_{j=2}^{+\infty} g(\tau_{j-1} + s, \tilde{\xi}_{\tau_{j-1}-}, \tilde{\xi}_{\tau_{j-1}}) 1_{\{\tau_{j-1} \leq t-s\}} \right) \pi^h(\xi_s^h, dy) \right], \end{aligned}$$

$$\begin{aligned}
&= \Pi_x^h \left[\int_0^t q(\xi_s^h) e_q(s) ds \int_E f(s, \xi_s^h, y) \tilde{\Pi}_y \left(\sum_{\tau_k \leq t-s} g(\tau_k + s, \tilde{\xi}_{\tau_k-}, \tilde{\xi}_{\tau_k}) \right) \pi^h(\xi_s^h, dy) \right] \\
&= \Pi_x^h \left[\int_0^t q(\xi_s^h) e_q(s) ds \int_E f(s, \xi_s^h, y) \tilde{\Pi}_y \left(\int_0^{t-s} q(\tilde{\xi}_r) dr \int_E g(r + s, \tilde{\xi}_r, z) \pi^h(\tilde{\xi}_r, dz) \right) \pi^h(\xi_s^h, dy) \right] \\
&= \tilde{\Pi}_x \left[f(\tau_1, \tilde{\xi}_{\tau_1-}, \tilde{\xi}_{\tau_1}) \tilde{\Pi}_{\tilde{\xi}_{\tau_1}} \left(\int_0^{t-s} q(\tilde{\xi}_r) dr \int_E g(r + s, \tilde{\xi}_r, z) \pi^h(\tilde{\xi}_r, dz) \right) \Big|_{s=\tau_1} 1_{\{\tau_1 \leq t\}} \right] \\
&= \tilde{\Pi}_x \left[f(\tau_1, \tilde{\xi}_{\tau_1-}, \tilde{\xi}_{\tau_1}) \int_{\tau_1}^t q(\tilde{\xi}_r) dr \int_E g(r, \tilde{\xi}_r, z) \pi^h(\tilde{\xi}_r, dz) 1_{\{\tau_1 \leq t\}} \right].
\end{aligned}$$

Similarly, by (4.15), Fubini's theorem, the strong Markov property of $\tilde{\xi}$, (4.7) and (4.10), we can prove by induction that for $i \geq 1$,

$$\begin{aligned}
&\sum_{j=i+1}^{+\infty} \tilde{\Pi}_x \left[f(\tau_i, \tilde{\xi}_{\tau_i-}, \tilde{\xi}_{\tau_i}) g(\tau_j, \tilde{\xi}_{\tau_j-}, \tilde{\xi}_{\tau_j}) 1_{\{\tau_j \leq t\}} \right] \\
&= \tilde{\Pi}_x \left[f(\tau_i, \tilde{\xi}_{\tau_i-}, \tilde{\xi}_{\tau_i}) \int_{\tau_i}^t q(\tilde{\xi}_r) dr \int_E g(r, \tilde{\xi}_r, z) \pi^h(\tilde{\xi}_r, dz) 1_{\{\tau_i \leq t\}} \right].
\end{aligned}$$

By this, Fubini's theorem, the strong Markov property of $\tilde{\xi}$ and (4.7), we get

$$\begin{aligned}
&\sum_{i=1}^{+\infty} \sum_{j=i+1}^{+\infty} \tilde{\Pi}_x \left[f(\tau_i, \tilde{\xi}_{\tau_i-}, \tilde{\xi}_{\tau_i}) g(\tau_j, \tilde{\xi}_{\tau_j-}, \tilde{\xi}_{\tau_j}) 1_{\{\tau_j \leq t\}} \right] \\
&= \tilde{\Pi}_x \left[\sum_{i=1}^{+\infty} f(\tau_i, \tilde{\xi}_{\tau_i-}, \tilde{\xi}_{\tau_i}) \int_{\tau_i}^t q(\tilde{\xi}_r) dr \int_E g(r, \tilde{\xi}_r, z) \pi^h(\tilde{\xi}_r, dz) 1_{\{\tau_i \leq t\}} \right] \\
&= \tilde{\Pi}_x \left[\sum_{\tau_i \leq t} f(\tau_i, \tilde{\xi}_{\tau_i-}, \tilde{\xi}_{\tau_i}) \int_{\tau_i}^t q(\tilde{\xi}_r) dr \int_E g(r, \tilde{\xi}_r, z) \pi^h(\tilde{\xi}_r, dz) \right] \\
&= \tilde{\Pi}_x \left[\sum_{\tau_i \leq t} f(\tau_i, \tilde{\xi}_{\tau_i-}, \tilde{\xi}_{\tau_i}) \tilde{\Pi}_{\tilde{\xi}_{\tau_i}} \left(\int_0^{t-s} q(\tilde{\xi}_r) dr \int_E g(s + r, \tilde{\xi}_r, z) \pi^h(\tilde{\xi}_r, dz) \right) \Big|_{s=\tau_i} \right] \\
&= \tilde{\Pi}_x \left[\int_0^t q(\tilde{\xi}_s) ds \int_E f(s, \tilde{\xi}_s, y) \tilde{\Pi}_y \left(\int_0^{t-s} q(\tilde{\xi}_r) dr \int_E g(s + r, \tilde{\xi}_r, z) \pi^h(\tilde{\xi}_r, dz) \right) \pi^h(\tilde{\xi}_s, dy) \right].
\end{aligned} \tag{4.16}$$

Combining (4.13) and (4.16), we arrive at (4.8). \square

4.2 Spine decomposition

In this section we work under Assumptions 0-2. Recall that the process $W_t^h(X)$ defined in Theorem 3.2 is a non-negative P_μ -martingale for every $\mu \in \mathcal{M}(E)$. We can define a new probability measure Q_μ for every $\mu \in \mathcal{M}(E)^0$ by the following formula:

$$dQ_\mu|_{\mathcal{F}_t} := \frac{1}{\langle h, \mu \rangle} W_t^h(X) dP_\mu \Big|_{\mathcal{F}_t} \quad \text{for all } t \geq 0.$$

It then follows from Proposition 2.1 that for any $f \in \mathcal{B}_b^+(E)$ and $t \geq 0$,

$$Q_\mu \left(e^{-\langle f, X_t \rangle} \right) = \frac{e^{\lambda_1 t}}{\langle h, \mu \rangle} P_\mu \left(\langle h, X_t \rangle e^{-\langle f, X_t \rangle} \right) = \frac{e^{\lambda_1 t}}{\langle h, \mu \rangle} e^{-\langle V_t f, \mu \rangle} \langle V_t^h f, \mu \rangle,$$

where $V_t^h f(x)$ is the unique locally bounded solution to (2.7) with initial value h . In this section we shall establish the spine decomposition of X under Q_μ .

Definition 4.4. *For every $\mu \in \mathcal{M}(E)$ and $x \in E$, there is a probability space with probability measure $\mathbb{P}_{\mu,x}$ that carries the following processes.*

- (i) $((\tilde{\xi}_t)_{t \geq 0}; \mathbb{P}_{\mu,x})$ is equal in law to $\tilde{\xi}$, a copy of the concatenation process starting from x ;
- (ii) $(n; \mathbb{P}_{\mu,x})$ is a random measure such that, given $\tilde{\xi}$ starting from x , n is a Poisson random measure which issues $\mathcal{M}(E)$ -valued processes $X^{n,t} := (X_s^{n,t})_{s \geq 0}$ at space-time points $(\tilde{\xi}_t, t)$ with rate

$$d\mathbb{N}_{\tilde{\xi}_t} \times 2b(\tilde{\xi}_t)dt.$$

Here for every $y \in E_+ = \{z \in E : b(z) > 0\}$, \mathbb{N}_y denotes the Kuznetsov measure on \mathcal{W}_0^+ corresponding to the (P_t, ϕ^L, ϕ^{NL}) -superprocess, while for $y \in E \setminus E_+$, \mathbb{N}_y denotes the null measure on \mathcal{W}_0^+ . Note that, given $\tilde{\xi}$, immigration happens only at space-time points

$(\tilde{\xi}_t, t)$ with $b(\tilde{\xi}_t) > 0$. Let D^n denote the almost surely countable set of immigration times, and $D_t^n := D^n \cap [0, t]$. Given $\tilde{\xi}$, the processes $\{X^{n,t} : t \in D^n\}$ are mutually independent.

- (iii) $(m; \mathbb{P}_{\mu,x})$ is a random measure such that, given $\tilde{\xi}$ starting from x , m is a Poisson random measure which issues $\mathcal{M}(E)$ -valued processes $X^{m,t} := (X_s^{m,t})_{s \geq 0}$ at space-time points $(\tilde{\xi}_t, t)$ with initial mass θ at rate

$$\theta \Pi^L(\tilde{\xi}_t, d\theta) \times dP_{\theta \delta_{\tilde{\xi}_t}} \times dt.$$

Here $P_{\theta \delta_x}$ denotes the law of the (P_t, ϕ^L, ϕ^{NL}) -superprocess starting from $\theta \delta_x$. Let D^m denote the almost surely countable set of immigration times, and $D_t^m := D^m \cap [0, t]$. Given $\tilde{\xi}$, the processes $\{X^{m,t} : t \in D^m\}$ are mutually independent, also independent of n and $\{X^{n,t} : t \in D^n\}$.

- (iv) $\{((X_s^{r,i})_{s \geq 0}; \mathbb{P}_{\mu,x}), i \geq 1\}$ is a family of $\mathcal{M}(E)$ -valued processes such that, given $\tilde{\xi}$ starting from x (including its revival times $\{\tau_i : i \geq 1\}$), $X^{r,i} := (X_s^{r,i})_{s \geq 0}$ is equal in law to $((X_s)_{s \geq 0}, P_{\pi_i})$ where P_{π_i} denotes the law of the (P_t, ϕ^L, ϕ^{NL}) -superprocess starting from $\pi_i(\cdot) := \Theta_i \pi(\tilde{\xi}_{\tau_i-}, \cdot)$ and Θ_i is a $[0, +\infty)$ -valued random variable with distribution $\eta(\tilde{\xi}_{\tau_i-}, d\theta)$ given by

$$\eta(x, d\theta) := \left(\frac{c(x)}{\gamma(x)} 1_A(x) + 1_{E \setminus A}(x) \right) \delta_0(d\theta) + \frac{1}{\gamma(x)} 1_A(x) 1_{(0, +\infty)}(\theta) \theta \Pi^{NL}(x, d\theta). \quad (4.17)$$

Moreover, given $\tilde{\xi}$ starting from x (including $\{\tau_i : i \geq 1\}$), $\{\Theta_i : i \geq 1\}$ are mutually independent, $\{X^{r,i} : i \geq 1\}$ are mutually independent, also independent of n, m , $\{X^{n,t} : t \in D^n\}$ and $\{X^{m,t} : t \in D^m\}$.

(v) $((X_t)_{t \geq 0}; \mathbb{P}_{\mu, x})$ is equal in law to $((X_t)_{t \geq 0}; \mathbb{P}_\mu)$, a copy of the (P_t, ϕ^L, ϕ^{NL}) -superprocess starting from μ . Moreover $((X_t)_{t \geq 0}; \mathbb{P}_{\mu, x})$ is independent of $\tilde{\xi}$, n , m and all the immigration processes.

We denote by

$$I_t^c := \sum_{s \in D_t^n} X_{t-s}^{n,s}, \quad I_t^d := \sum_{s \in D_t^m} X_{t-s}^{m,s} \quad \text{and} \quad I_t^r := \sum_{\tau_i \leq t} X_{t-\tau_i}^{r,i}$$

the continuous immigration, the discontinuous immigration and the revival-caused immigration, respectively. We define Γ_t by

$$\Gamma_t := X_t + I_t^c + I_t^d + I_t^r, \quad \forall t \geq 0.$$

The process $\tilde{\xi}$ is called the spine process, and the process $I_t := I_t^c + I_t^d + I_t^r$ is called the immigration process.

For any $\mu \in \mathcal{M}(E)$ and any measure ν on $(E, \mathcal{B}(E))$ with $0 < \langle h, \nu \rangle < +\infty$, we randomize the law $\mathbb{P}_{\mu, x}$ by replacing the deterministic choice of x with an E -valued random variable having distribution $h(x)\nu(dx)/\langle h, \nu \rangle$. We denote the resulting law by $\mathbb{P}_{\mu, \nu}$. That is to say,

$$\mathbb{P}_{\mu, \nu}(\cdot) := \frac{1}{\langle h, \nu \rangle} \int_E \mathbb{P}_{\mu, x}(\cdot) h(x) \nu(dx).$$

Clearly $\mathbb{P}_{\mu, \delta_x} = \mathbb{P}_{\mu, x}$. Since the laws of X and $(\tilde{\xi}, I)$ under $\mathbb{P}_{\mu, \nu}$ do not depend on ν and μ respectively, we sometimes write $\mathbb{P}_{\mu, \cdot}$ or $\mathbb{P}_{\cdot, \nu}$. For simplicity we also write \mathbb{P}_μ for $\mathbb{P}_{\mu, \mu}$. Here we take the convention that $\mathbb{P}_0(\Gamma_t = 0 \forall t \geq 0) = 1$.

For $s \geq 0$, define

$$\Lambda_s^m := \langle 1, X_0^{m,s} \rangle, \quad \text{if } s \in D^m \quad \text{and} \quad \Lambda_s^m := 0 \quad \text{elsewise.} \quad (4.18)$$

Then, given $\tilde{\xi}$, $\{\Lambda_s^m, s \geq 0\}$ is a Poisson point process with characteristic measure $\theta \Pi^L(\tilde{\xi}_s, d\theta)$. Let \mathcal{G} be the σ -field generated by $\tilde{\xi}$ (including $\{\tau_i : i \geq 1\}$), $\{\Theta_i : i \geq 1\}$, $\{D_t^m : t \geq 0\}$, $\{D_t^n : t \geq 0\}$, and $\{\Lambda_s^m, s \geq 0\}$.

Proposition 4.5. For $\mu \in \mathcal{M}(E)^0$, $f \in \mathcal{B}_b^+(E)$ and $t \geq 0$,

$$\mathbb{P}_\mu[\langle f, \Gamma_t \rangle | \mathcal{G}] = \langle \mathfrak{P}_t f, \mu \rangle + \sum_{s \in D_t^n} \mathfrak{P}_{t-s} f(\tilde{\xi}_s) + \sum_{s \in D_t^m} \Lambda_s^m \mathfrak{P}_{t-s} f(\tilde{\xi}_s) + \sum_{\tau_i \leq t} \Theta_i \pi(\tilde{\xi}_{\tau_i-}, \mathfrak{P}_{t-\tau_i} f) \quad \mathbb{P}_\mu\text{-a.s.} \quad (4.19)$$

Proof. By (2.9), we have for every $x \in E_+ = \{x \in E : b(x) > 0\}$, $f \in \mathcal{B}_b^+(E)$ and $t > 0$,

$$\mathbb{N}_x(\langle f, X_t \rangle) = \mathbb{P}_{\delta_x}(\langle f, X_t \rangle) = \mathfrak{P}_t f(x).$$

Let $D_t^r := \{\tau_i : \tau_i \leq t\}$. Then by the definition of Γ_t , under \mathbb{P}_μ ,

$$\begin{aligned} & \mathbb{P}_\mu[\langle f, \Gamma_t \rangle | \mathcal{G}] \\ &= \mathbb{P}_\mu(\langle f, X_t \rangle) + \sum_{s \in D_t^n} \mathbb{P}_\mu[\langle f, X_{t-s}^{n,s} \rangle | \mathcal{G}] + \sum_{s \in D_t^m} \mathbb{P}_\mu[\langle f, X_{t-s}^{m,s} \rangle | \mathcal{G}] + \sum_{s=\tau_i \in D_t^r} \mathbb{P}_\mu[\langle f, X_{t-s}^{r,i} \rangle | \mathcal{G}] \end{aligned}$$

$$\begin{aligned}
&= P_\mu(\langle f, X_t \rangle) + \sum_{s \in D_t^n} N_{\tilde{\xi}_s}(\langle f, X_{t-s} \rangle) + \sum_{s \in D_t^m} P_{\Lambda_s^m \delta_{\tilde{\xi}_s}}(\langle f, X_{t-s} \rangle) + \sum_{s=\tau_i \in D_t^r} P_{\pi_i}(\langle f, X_{t-s} \rangle) \\
&= \langle \mathfrak{P}_t f, \mu \rangle + \sum_{s \in D_t^n} \mathfrak{P}_{t-s} f(\tilde{\xi}_s) + \sum_{s \in D_t^m} \Lambda_s^m \mathfrak{P}_{t-s} f(\tilde{\xi}_s) + \sum_{s=\tau_i \in D_t^r} \Theta_i \pi(\tilde{\xi}_{s-}, \mathfrak{P}_{t-s} f).
\end{aligned}$$

□

The following is our main result on the spine decomposition of superprocesses with non-local branching mechanisms. Its proof will be given in the next subsection.

Theorem 4.6. *Suppose that Assumptions 0-2 hold. For every $\mu \in \mathcal{M}(E)^0$, the process $((\Gamma_t)_{t \geq 0}; \mathbb{P}_\mu)$ is Markovian and has the same law as $((X_t)_{t \geq 0}; \mathbb{Q}_\mu)$.*

Remark 4.7. In the case of purely local branching mechanism, the revival-caused immigration does not occur. To be more specific, in that case the spine runs as a copy of the h -transformed process ξ^h while only continuous and discontinuous immigration occur along the spine. The concatenating procedure and the revival-caused immigration are consequences of non-local branching. Similar phenomenon has been observed in [29] for multitype continuous-state branching processes and in [11] for multitype superdiffusions.

4.3 Proof of Theorem 4.6

In this subsection, we give the proof of Theorem 4.6. In order to do this, we prove a few lemmas first.

Lemma 4.8. *For every $x \in E$, $t \geq 0$ and $f \in \mathcal{B}_b^+(E)$,*

$$\mathbb{P}_{\cdot, x} \left[\exp \left(-\langle f, I_t^c + I_t^d \rangle \right) \mid \tilde{\xi}_s : 0 \leq s \leq t \right] = \exp \left(-\int_0^t \Phi(\tilde{\xi}_s, V_{t-s} f(\tilde{\xi}_s)) ds \right),$$

where $\Phi(x, \lambda) := 2b(x)\lambda + \int_{(0, +\infty)} \theta (1 - e^{-\lambda\theta}) \Pi^L(x, d\theta)$ for $x \in E$ and $\lambda \geq 0$.

Proof. This lemma follows from an argument which is almost identical to the one leading to (59) – (60) in [27]. We omit the details here. □

Lemma 4.9. *Suppose $f, l \in \mathcal{B}_b^+(E)$ and $(x, s) \mapsto g_s(x)$ is a non-negative locally bounded measurable function on $E \times [0, +\infty)$. For every $x \in E$ and $t > 0$, let*

$$e^{-w(x, t)} := \mathbb{P}_{\cdot, x} \left[\exp \left(-\int_0^t g_{t-s}(\tilde{\xi}_s) ds - \langle f, I_t^r \rangle - l(\tilde{\xi}_t) \right) \right].$$

Then $u(t, x) := e^{-\lambda_1 t} h(x) e^{-w(x, t)}$ satisfies the following integral equation:

$$\begin{aligned}
u(t, x) &= \Pi_x \left[e^{-l(\xi_t)} h(\xi_t) \right] \\
&\quad + \Pi_x \left[\int_0^t \left(\Phi(\xi_s, V_{t-s} f(\xi_s)) u(t-s, \xi_s) - \Psi(\xi_s, V_{t-s} f, u^{t-s}) - g_{t-s}(\xi_s) u(t-s, \xi_s) \right) ds \right],
\end{aligned} \tag{4.20}$$

where Ψ and Φ are defined in Proposition 2.1 and Lemma 4.8 respectively.

Proof. Following [20], it suffices to prove the result in the case when g does not depend on the time variable. Let τ_1 denote the first revival time of $\tilde{\xi}$. We have the following fundamental equation:

$$\begin{aligned} e^{-w(x,t)} &= \Pi_x^h \left[e_{q+g}(t) e^{-l(\xi_t^h)} \right] \\ &\quad + \Pi_x^h \left[\int_0^t q(\xi_s^h) e_{q+g}(s) \pi^h(\xi_s^h, e^{-w_{t-s}}) ds \int_{[0,+\infty)} e^{-\theta \pi(\xi_s^h, V_{t-s} f)} \eta(\xi_s^h, d\theta) \right]. \end{aligned}$$

The first term corresponds to the case when $\tau_1 \geq t$ and the second term corresponds to the case when the first revival happens at time $s \in (0, t)$. It then follows from Fubini's theorem and (3.12) that

$$\begin{aligned} &e^{-\lambda_1 t} h(x) e^{-w(x,t)} \\ &= \Pi_x \left[e_{a+g}(t) h(\xi_t) e^{-l(\xi_t)} \right] \\ &\quad + \Pi_x \left[\int_0^t e_{a+g}(s) q(\xi_s) h(\xi_s) \pi^h(\xi_s, e^{-\lambda_1(t-s)} e^{-w_{t-s}}) ds \int_{[0,+\infty)} e^{-\theta \pi(\xi_s, V_{t-s} f)} \eta(\xi_s, d\theta) \right]. \end{aligned} \tag{4.21}$$

We can continue the calculation in (4.21) by [31, Proposition 2.9] and (4.17) to get

$$\begin{aligned} u(t, x) &= e^{-\lambda_1 t} h(x) e^{-w(x,t)} \\ &= \Pi_x \left[h(\xi_t) e^{-l(\xi_t)} \right] \\ &\quad + \Pi_x \left[\int_0^t q(\xi_s) h(\xi_s) e^{-\lambda_1(t-s)} \pi^h(\xi_s, e^{-w_{t-s}}) ds \int_{[0,+\infty)} e^{-\theta \pi(\xi_s, V_{t-s} f)} \eta(\xi_s, d\theta) \right] \\ &\quad - \Pi_x \left[\int_0^t (a(\xi_s) + g(\xi_s)) e^{-\lambda_1(t-s)} h(\xi_s) e^{-w(\xi_s, t-s)} ds \right] \\ &= \Pi_x \left[h(\xi_t) e^{-l(\xi_t)} \right] \\ &\quad + \Pi_x \left[\int_0^t \pi(\xi_s, e^{-\lambda_1(t-s)} h e^{-w_{t-s}}) ds \left(c(\xi_s) + \int_{(0,+\infty)} r e^{-\theta \pi(\xi_s, V_{t-s} f)} \Pi^{NL}(\xi_s, d\theta) \right) \right] \\ &\quad - \Pi_x \left[\int_0^t (a(\xi_s) + g(\xi_s)) e^{-\lambda_1(t-s)} h(\xi_s) e^{-w(\xi_s, t-s)} ds \right]. \end{aligned}$$

This directly leads to (4.20). \square

Lemma 4.10. *For every $f, g \in \mathcal{B}_b^+(E)$, $\mu \in \mathcal{M}(E)$, $x \in E$ and $t \geq 0$,*

$$\mathbb{P}_{\mu, x} \left[\exp \left(-\langle f, \Gamma_t \rangle - g(\tilde{\xi}_t) \right) \right] = \frac{e^{\lambda_1 t}}{h(x)} e^{-\langle V_t f, \mu \rangle} V_t^{he^{-g}} f(x), \tag{4.22}$$

where $V_t^{he^{-g}} f(x)$ is the unique locally bounded solution to (2.7) with initial value $h e^{-g}$.

Proof. Recall that $(X; \mathbb{P}_{\mu, x})$ is independent of $\tilde{\xi}$ and all the immigration processes. Moreover, given $\tilde{\xi}$ (including $\{\tau_i : i \geq 1\}$), I^r is independent of I^c and I^d . It then follows from Lemma 4.8 that

$$\mathbb{P}_{\mu, x} \left[\exp \left(-\langle f, \Gamma_t \rangle - g(\tilde{\xi}_t) \right) \right]$$

$$\begin{aligned}
&= \mathbb{P}_{\mu,x} \left[\exp \left(-\langle f, X_t \rangle - \langle f, I_t^c + I_t^d \rangle - \langle f, I_t^r \rangle - g(\tilde{\xi}_t) \right) \right] \\
&= \mathbb{P}_{\mu,x} \left[e^{-\langle f, X_t \rangle} \right] \mathbb{P}_{\mu,x} \left\{ e^{-g(\tilde{\xi}_t)} \mathbb{P}_{\mu,x} \left[\exp(-\langle f, I_t^r \rangle) \mid \{\tilde{\xi}_s : 0 \leq s \leq t\} \cup \{\tau_i : \tau_i \leq t\} \right] \right. \\
&\quad \left. \mathbb{P}_{\mu,x} \left[\exp \left(-\langle f, I_t^c + I_t^d \rangle \right) \mid \tilde{\xi}_s : 0 \leq s \leq t \right] \right\} \\
&= e^{-\langle V_t f, \mu \rangle} \mathbb{P}_{\mu,x} \left[\exp \left(-\int_0^t \Phi(\tilde{\xi}_s, V_{t-s} f(\tilde{\xi}_s)) ds - \langle f, I_t^r \rangle - g(\tilde{\xi}_t) \right) \right]. \tag{4.23}
\end{aligned}$$

Let $v(t, x) := e^{-\lambda_1 t} h(x) \mathbb{P}_{\mu,x} \left[\exp \left(-\int_0^t \Phi(\tilde{\xi}_s, V_{t-s} f(\tilde{\xi}_s)) ds - \langle f, I_t^r \rangle - g(\tilde{\xi}_t) \right) \right]$. One can easily verify that $(x, s) \mapsto g_s(x) := \Phi(x, V_s f(x))$ is a locally bounded function. Thus by Lemma 4.9, $v(t, x)$ is a locally bounded solution to the equation (2.7) with initial value $h e^{-g}$. By the uniqueness of such solution, we have $v(t, x) = V_t^{h e^{-g}} f(x)$. This and (4.23) lead to (4.22). \square

Proof of Theorem 4.6:

The proof is inspired by the calculations in the proof of [20, Theorem 3.2]. First we claim that for every $\mu \in \mathcal{M}(E)^0$, $((\Gamma_t)_{t \geq 0}; \mathbb{P}_\mu)$ has the same one dimensional distribution as $((X_t)_{t \geq 0}; \mathbb{Q}_\mu)$. This would follow if for every $f \in \mathcal{B}_b^+(E)$ and every $t \geq 0$,

$$\mathbb{P}_\mu \left(e^{-\langle f, \Gamma_t \rangle} \right) = \mathbb{Q}_\mu \left(e^{-\langle f, X_t \rangle} \right). \tag{4.24}$$

By the definition of \mathbb{Q}_μ and Proposition 2.1,

$$\mathbb{Q}_\mu \left(e^{-\langle f, X_t \rangle} \right) = \frac{e^{\lambda_1 t}}{\langle h, \mu \rangle} \mathbb{P}_\mu \left[\langle h, X_t \rangle e^{-\langle f, X_t \rangle} \right] = \frac{e^{\lambda_1 t}}{\langle h, \mu \rangle} e^{-\langle V_t f, \mu \rangle} \langle V_t^h f, \mu \rangle, \tag{4.25}$$

where $V_t^h f(x)$ is the unique locally bounded solution to (2.7) with initial value h . By Lemma 4.10, we have $\mathbb{P}_{\mu,x} [\exp(-\langle f, \Gamma_t \rangle)] = \exp(\lambda_1 t - \langle V_t f, \mu \rangle) h(x)^{-1} V_t^h f(x)$. Thus

$$\begin{aligned}
\mathbb{P}_\mu \left(e^{-\langle f, \Gamma_t \rangle} \right) &= \frac{1}{\langle h, \mu \rangle} \int_E \mathbb{P}_{\mu,x} \left(e^{-\langle f, \Gamma_t \rangle} \right) h(x) \mu(dx) \\
&= \frac{e^{\lambda_1 t}}{\langle h, \mu \rangle} e^{-\langle V_t f, \mu \rangle} \langle V_t^h f, \mu \rangle. \tag{4.26}
\end{aligned}$$

Combining (4.25) and (4.26), we get (4.24). It follows that for every $\mu \in \mathcal{M}(E)^0$,

$$\mathbb{P}_\mu(\Gamma_t = 0) = \mathbb{Q}_\mu(X_t = 0) = \frac{1}{\langle h, \mu \rangle} \mathbb{P}_\mu \left(W_t^h(X); X_t = 0 \right) = 0 \quad \forall t > 0. \tag{4.27}$$

It remains to prove the Markov property of $((\Gamma_t)_{t \geq 0}; \mathbb{P}_\mu)$. To do this, we apply [20, Lemma 3.3] here. Recall that $E_\partial = E \cup \{\partial\}$ where ∂ is a cemetery point. We can extend the probability measure $\mathbb{P}_{\mu,x}$ onto $\mu \times \{\partial\}$ by defining that $\mathbb{P}_{\mu,\partial}(\tilde{\xi}_t = \partial, I_t = 0 \forall t \geq 0) = 1$ for all $\mu \in \mathcal{M}(E)$. In the remainder of this proof, we call \mathfrak{J} a Markov kernel if \mathfrak{J} is a map from the measurable space (S, \mathcal{S}) to the measurable space (S', \mathcal{S}') such that for every $y \in S$, $\mathfrak{J}(y, \cdot)$ is a probability measure on (S', \mathcal{S}') , and for every $B \in \mathcal{S}'$, $\mathfrak{J}(\cdot, B) \in b\mathcal{S}$ the space of bounded measurable functions on S . The kernel \mathfrak{J} will also be viewed as an operator taking $f \in b\mathcal{S}'$ to $\mathfrak{J}f \in b\mathcal{S}$ where $\mathfrak{J}f(y) := \int_{S'} f(z) \mathfrak{J}(y, dz)$.

Clearly $((Z_t)_{t \geq 0} := ((\Gamma_t, \tilde{\xi}_t)_{t \geq 0}; \mathbb{P}_{\mu,x})$ is a Markov process on $\mathcal{M}(E) \times E_\partial$. Denote by S_t the transition semigroup of Z_t , by \mathfrak{K} the Markov kernel from $\mathcal{M}(E) \times E_\partial$ to $\mathcal{M}(E)$ induced by the

projection from $\mathcal{M}(E) \times E_\partial$ onto $\mathcal{M}(E)$, and by \mathfrak{Q} the Markov kernel from $\mathcal{M}(E)$ to $\mathcal{M}(E) \times E_\partial$ given by

$$\mathfrak{Q}(\nu_1, d(\nu_2 \times x)) := 1_{\{\nu_1 \neq 0\}} \delta_{\nu_1}(d\nu_2) \times 1_E(x) \frac{h(x)\nu_1(dx)}{\langle h, \nu_1 \rangle} + 1_{\{\nu_1=0\}} \delta_0(d\nu_2) \times \delta_\partial(dx).$$

Let $R_t := \mathfrak{Q}S_t\mathfrak{K}$ for $t \geq 0$. One can easily verify that $\mathfrak{Q}\mathfrak{K}$ is the identical kernel on $\mathcal{M}(E)$ and $R_t(\nu_1, d\nu_2) = \mathbb{P}_{\nu_1}(\Gamma_t \in d\nu_2)$ for all $\nu_1 \in \mathcal{M}(E)$. By [20, Lemma 3.3], $((\Gamma_t)_{t \geq 0}; \mathbb{P}_\mu)$ is Markovian as long as $\mathfrak{Q}S_t = R_t\mathfrak{Q}$. This would follow if for all $f, g \in \mathcal{B}_b^+(E)$ and $\nu_1 \in \mathcal{M}(E)$,

$$\begin{aligned} & \int_{\mathcal{M}(E)} \int_{\mathcal{M}(E) \times E_\partial} e^{-\langle f, \nu_3 \rangle - g(y)} \mathfrak{Q}(\nu_2, d(\nu_3 \times y)) R_t(\nu_1, d\nu_2) \\ &= \int_{\mathcal{M}(E) \times E_\partial} \int_{\mathcal{M}(E) \times E_\partial} e^{-\langle f, \nu_3 \rangle - g(y)} S_t(\nu_2 \times x, d(\nu_3 \times y)) \mathfrak{Q}(\nu_1, d(\nu_2 \times x)). \end{aligned} \quad (4.28)$$

By the above definitions, we have

$$\text{LHS of (4.28)} = \mathbb{P}_{\nu_1} \left[e^{-\langle f, \Gamma_t \rangle} \frac{\langle h e^{-g}, \Gamma_t \rangle}{\langle h, \Gamma_t \rangle} 1_{\{\Gamma_t \neq 0\}} \right] + \mathbb{P}_{\nu_1}(\Gamma_t = 0)$$

and

$$\text{RHS of (4.28)} = \mathbb{P}_{\nu_1} \left[e^{-\langle f, \Gamma_t \rangle - g(\tilde{\xi}_t)} \right] 1_{\{\nu_1 \neq 0\}} + 1_{\{\nu_1=0\}}.$$

In view of (4.27), to show (4.28), it suffices to show that for any $\mu \in \mathcal{M}(E)^0$ and $f, g \in \mathcal{B}_b^+(E)$,

$$\mathbb{P}_\mu \left[e^{-\langle f, \Gamma_t \rangle - g(\tilde{\xi}_t)} \right] = \mathbb{P}_\mu \left[e^{-\langle f, \Gamma_t \rangle} \frac{\langle h e^{-g}, \Gamma_t \rangle}{\langle h, \Gamma_t \rangle} 1_{\{\Gamma_t \neq 0\}} \right]. \quad (4.29)$$

It follows from Lemma 4.10 that

$$\begin{aligned} \mathbb{P}_\mu \left[e^{-\langle f, \Gamma_t \rangle - g(\tilde{\xi}_t)} \right] &= \frac{1}{\langle h, \mu \rangle} \int_E \mathbb{P}_{\mu, x} \left[e^{-\langle f, \Gamma_t \rangle - g(\tilde{\xi}_t)} \right] h(x) \mu(dx) \\ &= \frac{e^{\lambda_1 t}}{\langle h, \mu \rangle} e^{-\langle V_t f, \mu \rangle} \langle V_t^{h e^{-g}} f, \mu \rangle, \end{aligned} \quad (4.30)$$

where $V_t^{h e^{-g}} f(x)$ is the unique locally bounded solution to (2.7) with initial value $h e^{-g}$. On the other hand, since $(\Gamma_t, \mathbb{P}_\mu)$ and (X_t, \mathbb{Q}_μ) are identically distributed for each $t \geq 0$, we have by the definition of \mathbb{Q}_μ and Proposition 2.1 that

$$\begin{aligned} \mathbb{P}_\mu \left[e^{-\langle f, \Gamma_t \rangle} \frac{\langle h e^{-g}, \Gamma_t \rangle}{\langle h, \Gamma_t \rangle} 1_{\{\Gamma_t \neq 0\}} \right] &= \mathbb{Q}_\mu \left[e^{-\langle f, X_t \rangle} \frac{\langle h e^{-g}, X_t \rangle}{\langle h, X_t \rangle} 1_{\{X_t \neq 0\}} \right] \\ &= \frac{e^{\lambda_1 t}}{\langle h, \mu \rangle} \mathbb{P}_\mu \left[e^{-\langle f, X_t \rangle} \langle h e^{-g}, X_t \rangle \right] \\ &= \frac{e^{\lambda_1 t}}{\langle h, \mu \rangle} e^{-\langle V_t f, \mu \rangle} \langle V_t^{h e^{-g}} f, \mu \rangle. \end{aligned} \quad (4.31)$$

Combining (4.30) and (4.31), we get (4.29). The proof is now complete. \square

5 Sufficient condition for non-degenerate martingale limit

In this section, we will give sufficient conditions for the fundamental martingale to have a non-degenerate limit. We start with an assumption.

Assumption 3.

- (i) $a(x), \gamma(x) \in L^2(E, m)$.
- (ii) $(1_A \pi(\cdot, h), \hat{h}) < +\infty$.
- (iii) $x \mapsto \pi(x, h)/h$ is bounded from above on A .

It is easy to see that Assumption 3.(iii) implies Assumption 3.(ii). In this section we will use the first two items of this assumption. In the next section we will use items (i) and (iii) of this assumption.

The following theorem is the main result of this section.

Theorem 5.1. *Suppose Assumptions 0-2 and 3.(i)-(ii) hold and that*

$$\left(\int_{(0, +\infty)} rh(\cdot) \log^+(rh(\cdot)) \Pi^L(\cdot, dr), \hat{h} \right) + \left(1_A(\cdot) \int_{(0, +\infty)} r\pi(\cdot, h) \log^+(r\pi(\cdot, h)) \Pi^{NL}(\cdot, dr), \hat{h} \right) < +\infty. \quad (5.1)$$

For every $\mu \in \mathcal{M}(E)$, the limit $W_\infty^h(X) := \lim_{t \rightarrow +\infty} W_t^h(X)$ exists P_μ -a.s. Furthermore

- (i) if $\lambda_1 < 0$, then the martingale $W_t^h(X)$ converges to $W_\infty^h(X)$ as $t \rightarrow \infty$ P_μ -a.s. and in $L^1(P_\mu)$, and $W_\infty^h(X)$ is non-degenerate in the sense that $P_\mu(W_\infty^h(X) > 0) > 0$ for $\mu \neq 0$;
- (ii) if $\lambda_1 > 0$, then $W_\infty^h(X) = 0$ P_μ -a.s.

In the remainder of this section we will assume Assumptions 0-2 hold. Additional conditions used are stated explicitly. To prove Theorem 5.1, we need a few lemmas.

Proposition 5.2. *Suppose Assumption 3.(i) holds. For all $f \in \mathcal{B}_b(E) \cap L^2(E, m)$ and $s, t \in (0, +\infty)$,*

$$\lim_{s \rightarrow t} \mathfrak{P}_s f = \mathfrak{P}_t f \quad \text{in } L^2(E, m). \quad (5.2)$$

Moreover,

$$\mathfrak{P}_t f = T_t f [m] \quad \text{for all } t > 0. \quad (5.3)$$

Proof. Fix $f \in \mathcal{B}_b(E) \cap L^2(E, m)$. We first prove (5.2). Without loss of generality, we assume $0 < s < t < +\infty$. Let $F_r(x) := -a(x)\mathfrak{P}_r f(x) + \gamma(x, \mathfrak{P}_r f)$. We have shown in the argument below (3.4) that $\|\mathfrak{P}_r f\|_\infty \leq e^{c_1 r} \|f\|_\infty$ for some constant $c_1 > 0$. Thus by definition, $|F_r(x)| \leq (|a(x)| + \gamma(x)) \|\mathfrak{P}_r f\|_\infty \leq e^{c_1 r} \|f\|_\infty (|a(x)| + \gamma(x))$. Clearly by the boundedness of $a(x)$ and $\gamma(x)$, $(x, r) \mapsto F_r(x)$ is locally bounded on $E \times [0, +\infty)$ and by Assumption 3.(i), $x \mapsto F_r(x) \in \mathcal{B}_b(E) \cap L^2(E, m)$. By (2.6), we have

$$\mathfrak{P}_t f(x) - \mathfrak{P}_s f(x)$$

$$= P_t f(x) - P_s f(x) + \Pi_x \left[\int_s^t F_r(\xi_{t-r}) dr \right] + \Pi_x \left[\int_0^s F_r(\xi_{t-r}) - F_r(\xi_{s-r}) dr \right]. \quad (5.4)$$

Recall that $\{P_t : t \geq 0\}$ is a strongly continuous contraction semigroup on $L^2(E, m)$. Thus

$$\|P_t f - P_s f\|_{L^2(E, m)} = \|P_s (P_{t-s} f - f)\|_{L^2(E, m)} \leq \|P_{t-s} f - f\|_{L^2(E, m)} \rightarrow 0 \quad (5.5)$$

as $s \rightarrow t$. Note that

$$\left| \Pi_x \left[\int_s^t F_r(\xi_{t-r}) dr \right] \right| \leq \int_s^t |\Pi_x [F_r(\xi_{t-r})]| dr = \int_s^t |P_{t-r} F_r(x)| dr. \quad (5.6)$$

We have by Minkowski's integral inequality and the contractivity of P_t that

$$\begin{aligned} \left\| \int_s^t |P_{t-r} F_r| dr \right\|_{L^2(E, m)} &\leq \int_s^t \|P_{t-r} F_r\|_{L^2(E, m)} dr \\ &\leq \int_s^t \|F_r\|_{L^2(E, m)} dr \\ &\leq \|f\|_\infty (\|a\|_{L^2(E, m)} + \|\gamma\|_{L^2(E, m)}) \int_s^t e^{c_1 r} dr \rightarrow 0 \quad \text{as } s \rightarrow t. \end{aligned}$$

This together with (5.6) implies that

$$\lim_{s \rightarrow t} \left\| \Pi_x \left[\int_s^t F_r(\xi_{t-r}) dr \right] \right\|_{L^2(E, m)} = 0. \quad (5.7)$$

Note that by the Markov property of ξ ,

$$\begin{aligned} \left| \Pi_x \left[\int_0^s F_r(\xi_{t-r}) - F_r(\xi_{s-r}) dr \right] \right| &\leq \int_0^s \Pi_x [|\Pi_{\xi_{s-r}}(F_r(\xi_{t-s})) - \Pi_{\xi_{s-r}}(F_r(\xi_0))|] dr \\ &= \int_0^s P_{s-r} (|P_{t-s} F_r - F_r|)(x) dr. \end{aligned} \quad (5.8)$$

It follows from the strong continuity and contractivity of the semigroup $\{P_t : t \geq 0\}$ that

$$\lim_{s \rightarrow t} \|P_{t-s} F_r - F_r\|_{L^2(E, m)} = 0$$

and

$$\|P_{t-s} F_r - F_r\|_{L^2(E, m)} \leq 2\|F_r\|_{L^2(E, m)} \leq 2e^{c_1 r} \|f\|_\infty (\|a\|_{L^2(E, m)} + \|\gamma\|_{L^2(E, m)}).$$

Thus by Minkowski's integral inequality and the dominated convergence theorem, we have

$$\begin{aligned} \left\| \int_0^s P_{s-r} (|P_{t-s} F_r - F_r|) dr \right\|_{L^2(E, m)} &\leq \int_0^s \|P_{s-r} (|P_{t-s} F_r - F_r|)\|_{L^2(E, m)} dr \\ &\leq \int_0^s \|P_{t-s} F_r - F_r\|_{L^2(E, m)} dr \rightarrow 0 \quad \text{as } s \rightarrow t. \end{aligned}$$

This together with (5.8) implies that

$$\lim_{s \rightarrow t} \left\| \Pi_x \left[\int_0^s F_r(\xi_{t-r}) - F_r(\xi_{s-r}) dr \right] \right\|_{L^2(E, m)} = 0. \quad (5.9)$$

Combining (5.4)–(5.9), we arrive at (5.2). To prove (5.3), it suffices to prove that for every $t > 0$ and every $g \in L^2(E, m)$,

$$\int_E \mathfrak{P}_t f(x) g(x) m(dx) = \int_E T_t f(x) g(x) m(dx). \quad (5.10)$$

Note that by Hölder's inequality and (5.2), for $s, t \in (0, +\infty)$,

$$\left| \int_E \mathfrak{P}_t f(x) g(x) m(dx) - \int_E \mathfrak{P}_s f(x) g(x) m(dx) \right| \leq \|\mathfrak{P}_t f - \mathfrak{P}_s f\|_{L^2(E, m)} \|g\|_{L^2(E, m)} \rightarrow 0$$

as $s \rightarrow t$. This implies $t \mapsto \int_E \mathfrak{P}_t f(x) g(x) m(dx)$ is a continuous function on $(0, +\infty)$. Similarly using the strong continuity of $\{T_t : t \geq 0\}$ on $L^2(E, m)$, one can prove that $t \mapsto \int_E T_t f(x) g(x) m(dx)$ is also a continuous function on $(0, +\infty)$. By taking Laplace transform of $\int_E \mathfrak{P}_t f(x) g(x) m(dx)$ (respectively, $\int_E T_t f(x) g(x) m(dx)$), we get $\int_E R_\alpha f(x) g(x) m(dx)$ (respectively, $\int_E U_\alpha f(x) g(x) m(dx)$). It has been shown in the argument below (3.4) that under Assumption 3.(i), $R_\alpha f = U_\alpha f [m]$ for α sufficiently large. So the Laplace transforms of both sides of (5.10) are identical for α sufficiently large. Hence (5.10) follows from Post's inversion theorem for Laplace transforms. \square

Proposition 5.3. *Under the assumptions of Proposition 5.2, the measure ρ is an invariant probability measure for $\{\tilde{P}_t : t \geq 0\}$, i.e., for every $t \geq 0$ and $f \in \mathcal{B}^+(E)$,*

$$\int_E \tilde{P}_t f(x) \rho(dx) = \int_E f(x) \rho(dx). \quad (5.11)$$

Proof. By the monotone convergence theorem, we only need to prove (5.11) for $f \in \mathcal{B}_b^+(E)$. Clearly $fh \in \mathcal{B}_b^+(E) \cap L^2(E, m)$. It follows by (4.3), (5.3) and (3.2) that

$$\begin{aligned} \int_E \tilde{P}_t f(x) \rho(dx) &= \int_E e^{\lambda_1 t} \mathfrak{P}_t(fh)(x) \hat{h}(x) m(dx) \\ &= \int_E e^{\lambda_1 t} T_t(fh)(x) \hat{h}(x) m(dx) \\ &= \int_E e^{\lambda_1 t} f(x) h(x) \hat{T}_t \hat{h}(x) m(dx) \\ &= \int_E f(x) \rho(dx). \end{aligned}$$

\square

Lemma 5.4. *The function $g(x) := h(x)^{-1} P_{\delta_x} [W_\infty^h(X)]$ satisfies that*

$$P_\mu [W_\infty^h(X)] = \langle gh, \mu \rangle \quad \text{for all } \mu \in \mathcal{M}(E). \quad (5.12)$$

Moreover,

$$\tilde{P}_t g(x) = g(x) \quad \text{for all } t \geq 0 \text{ and } x \in E. \quad (5.13)$$

Proof. To prove the first claim, we note that for an arbitrary constant $\lambda > 0$, by the bounded convergence theorem,

$$\begin{aligned} \mathbb{P}_\mu \left[\exp \left(-\lambda W_\infty^h(X) \right) \right] &= \lim_{t \rightarrow +\infty} \mathbb{P}_\mu \left[\exp \left(-\lambda W_t^h(X) \right) \right] \\ &= \lim_{t \rightarrow +\infty} \exp \left(-\langle l_\lambda(t, \cdot), \mu \rangle \right) \\ &= \exp \left(-\lim_{t \rightarrow +\infty} \langle l_\lambda(t, \cdot), \mu \rangle \right), \end{aligned} \quad (5.14)$$

where $l_\lambda(t, x) := -\log \mathbb{P}_{\delta_x} \left[\exp \left(-\lambda W_t^h(X) \right) \right]$. Let

$$l_\lambda(x) := \lim_{t \rightarrow +\infty} l_\lambda(t, x) = -\log \mathbb{P}_{\delta_x} \left[\exp \left(-\lambda W_\infty^h(X) \right) \right].$$

We have by Jensen's inequality that

$$l_\lambda(t, x) \leq \lambda \mathbb{P}_{\delta_x} \left(W_t^h(X) \right) = \lambda e^{\lambda_1 t} \mathfrak{P}_t h(x) = \lambda h(x) \quad \text{for all } x \in E, t \geq 0.$$

Hence $l_\lambda(x) \leq \lambda h(x)$ for all $x \in E$. This together with (5.14) and the dominated convergence theorem yields that

$$\mathbb{P}_\mu \left[\exp \left(-\lambda W_\infty^h(X) \right) \right] = e^{-\langle l_\lambda, \mu \rangle}. \quad (5.15)$$

Thus we get (5.12) by differentiating both sides of (5.15) with respect to λ and then letting $\lambda \downarrow 0$. Note that $0 \leq g \leq 1$ by Fatou's lemma. By the Markov property of X and (5.12), we have for any $t \geq 0$ and $x \in E$,

$$\begin{aligned} g(x) &= \frac{1}{h(x)} \mathbb{P}_{\delta_x} \left[\lim_{s \rightarrow +\infty} e^{\lambda_1(t+s)} \langle h, X_{t+s} \rangle \right] \\ &= \frac{e^{\lambda_1 t}}{h(x)} \mathbb{P}_{\delta_x} \left[\mathbb{P}_{X_t} \left(\lim_{s \rightarrow +\infty} W_s^h(X) \right) \right] = \frac{e^{\lambda_1 t}}{h(x)} \mathbb{P}_{\delta_x} \left[\mathbb{P}_{X_t} \left(W_\infty^h(X) \right) \right] \\ &= \frac{e^{\lambda_1 t}}{h(x)} \mathbb{P}_{\delta_x} [\langle gh, X_t \rangle] = \frac{e^{\lambda_1 t}}{h(x)} \mathfrak{P}_t(gh)(x) = \tilde{P}_t g(x). \end{aligned}$$

Here we used (4.3) in the last equality. □

Lemma 5.5. *Suppose Assumptions 0–2 and 3.(i) hold. Let Λ_s^m be as defined in (4.18). If condition (5.1) holds, then for m -almost every $x \in E$,*

$$\lim_{D^m \ni s \rightarrow +\infty} \frac{\log^+(\Lambda_s^m h(\tilde{\xi}_s))}{s} = \lim_{i \rightarrow +\infty} \frac{\log^+ \Theta_i \pi(\tilde{\xi}_{\tau_i-}, h)}{\tau_i} = 0 \quad \mathbb{P}_{\cdot, x} \text{-a.s.} \quad (5.16)$$

Proof. To prove (5.16), it suffices to prove that for any $\varepsilon > 0$ sufficiently small,

$$\mathbb{P}_{\cdot, x} \left(\sum_{s \in D^m} 1_{\{\Lambda_s^m h(\tilde{\xi}_s) > e^{\varepsilon s}\}} = +\infty \right) = 0 \text{ and } \mathbb{P}_{\cdot, x} \left(\sum_{i=1}^{+\infty} 1_{\{\Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) > e^{\varepsilon \tau_i}\}} = +\infty \right) = 0. \quad (5.17)$$

For an arbitrary $B \in \mathcal{B}(E)$ with $0 < m(B) < +\infty$, let $\mu_B(dx) := \hat{h}(x) 1_B(x) m(dx)$. Clearly $\mu_B \in \mathcal{M}(E)^0$. Recall that given $\tilde{\xi}$, $\{\Lambda_s^m : s \geq 0\}$ is a Poisson point process with characteristic

measure $\lambda \Pi^L(\tilde{\xi}_s, d\lambda)$. Thus by Fubini's theorem and the fact that $\rho(dx) = h(x)\hat{h}(x)m(dx)$ is an invariant measure for \tilde{P}_t , we have

$$\begin{aligned}
& \mathbb{P}_{\mu_B} \left(\sum_{s \in D^m} 1_{\{\Lambda_s^m h(\tilde{\xi}_s) > e^{\varepsilon s}\}} \right) \\
&= \mathbb{P}_{\mu_B} \left(\int_0^{+\infty} \int_{(0,+\infty)} \lambda 1_{\{\lambda h(\tilde{\xi}_s) > e^{\varepsilon s}\}} \Pi^L(\tilde{\xi}_s, d\lambda) ds \right) \\
&= \frac{1}{\langle h, \mu_B \rangle} \int_E \mathbb{P}_{\mu_B, x} \left(\int_0^{+\infty} \int_{(0,+\infty)} \lambda 1_{\{\lambda h(\tilde{\xi}_s) > e^{\varepsilon s}\}} \Pi^L(\tilde{\xi}_s, d\lambda) ds \right) 1_B(x) \rho(dx) \\
&\leq \frac{1}{\langle h, \mu_B \rangle} \int_0^{+\infty} ds \int_E \rho(dx) \int_{(0,+\infty)} \lambda 1_{\{\lambda h(x) > e^{\varepsilon s}\}} \Pi^L(x, d\lambda) \\
&= \frac{1}{\langle h, \mu_B \rangle} \left(\int_{(0,+\infty)} \lambda \Pi^L(\cdot, d\lambda) \int_0^{\log^+ \lambda h(x)/\varepsilon} ds, h\hat{h} \right) \\
&= \frac{1}{\varepsilon \langle h, \mu_B \rangle} \left(\int_{(0,+\infty)} \lambda h(\cdot) \log^+(\lambda h(\cdot)) \Pi^L(\cdot, d\lambda), \hat{h} \right). \tag{5.18}
\end{aligned}$$

The right hand side of (5.18) is finite by (5.1). Thus we get $\mathbb{P}_{\mu_B} \left(\sum_{s \in D^m} 1_{\{\Lambda_s^m h(\tilde{\xi}_s) > e^{\varepsilon s}\}} < +\infty \right) = 1$. Note that

$$\mathbb{P}_{\mu_B} \left(\sum_{s \in D^m} 1_{\{\Lambda_s^m h(\tilde{\xi}_s) > e^{\varepsilon s}\}} < +\infty \right) = \rho(B)^{-1} \int_B \mathbb{P}_{\cdot, x} \left(\sum_{s \in D^m} 1_{\{\Lambda_s^m h(\tilde{\xi}_s) > e^{\varepsilon s}\}} < +\infty \right) \rho(dx).$$

Thus $\mathbb{P}_{\cdot, x} \left(\sum_{s \in D^m} 1_{\{\Lambda_s^m h(\tilde{\xi}_s) > e^{\varepsilon s}\}} < +\infty \right) = 1$ for m -almost every $x \in B$. Since B is arbitrary, the first equality of (5.17) holds for m -almost every $x \in E$.

Recall that given $\tilde{\xi}$ (including $\{\tau_i : i \geq 1\}$), Θ_i is distributed as $\eta(\tilde{\xi}_{\tau_i-}, d\theta)$ given by (4.17). Thus by Fubini's theorem and (4.7),

$$\begin{aligned}
& \mathbb{P}_{\mu_B} \left(\sum_{i=1}^{+\infty} 1_{\{\Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) > e^{\varepsilon \tau_i}\}} \right) \\
&= \mathbb{P}_{\mu_B} \left[\sum_{i=1}^{+\infty} \int_{\theta \pi(\tilde{\xi}_{\tau_i-}, h) > e^{\varepsilon \tau_i}} \eta(\tilde{\xi}_{\tau_i-}, d\theta) \right] \\
&= \mathbb{P}_{\mu_B} \left[\sum_{i=1}^{+\infty} \frac{1}{\gamma(\tilde{\xi}_{\tau_i-})} 1_A(\tilde{\xi}_{\tau_i-}) \int_{\theta \pi(\tilde{\xi}_{\tau_i-}, h) > e^{\varepsilon \tau_i}} \theta \Pi^{NL}(\tilde{\xi}_{\tau_i-}, d\theta) \right] \\
&= \mathbb{P}_{\mu_B} \left[\int_0^{+\infty} q(\tilde{\xi}_s) \frac{1}{\gamma(\tilde{\xi}_s)} 1_A(\tilde{\xi}_s) ds \int_{\theta \pi(\tilde{\xi}_s, h) > e^{\varepsilon s}} \theta \Pi^{NL}(\tilde{\xi}_s, d\theta) \right] \\
&= \frac{1}{\langle h, \mu_B \rangle} \int_0^{+\infty} ds \int_E \mathbb{P}_{\mu, x} \left[\frac{\pi(\tilde{\xi}_s, h)}{h(\tilde{\xi}_s)} 1_A(\tilde{\xi}_s) \int_{\theta \pi(\tilde{\xi}_s, h) > e^{\varepsilon s}} \theta \Pi^{NL}(\tilde{\xi}_s, d\theta) \right] 1_B(x) \rho(dx) \\
&\leq \frac{1}{\langle h, \mu_B \rangle} \int_0^{+\infty} ds \int_E 1_A(x) \pi(x, h) \hat{h}(x) m(dx) \int_{\theta \pi(x, h) > e^{\varepsilon s}} \theta \Pi^{NL}(x, d\theta)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\langle h, \mu_B \rangle} \int_E 1_A(x) \pi(x, h) \hat{h}(x) m(dx) \int_{(0, +\infty)} \theta \Pi^{NL}(x, d\theta) \int_0^{\frac{\log^+(\theta \pi(x, h))}{\varepsilon}} ds \\
&= \frac{1}{\varepsilon \langle h, \mu_B \rangle} \left(\pi(\cdot, h) \int_{(0, +\infty)} \theta \log^+(\theta \pi(\cdot, h)) \Pi^{NL}(\cdot, d\theta), 1_A \hat{h} \right). \tag{5.19}
\end{aligned}$$

The right hand side of (5.19) is finite by (5.1). Thus we get $\mathbb{P}_{\mu_B} \left(\sum_{i=1}^{+\infty} 1_{\{\Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) > e^{\varepsilon \tau_i}\}} < +\infty \right) = 1$. Using an argument similar to that at the end of the first paragraph, one can prove that the second equality of (5.17) holds for m -almost every $x \in E$. \square

Proof of Theorem 5.1:

(i) Suppose $\lambda_1 < 0$. Without loss of generality, we assume $\mu \in \mathcal{M}(E)^0$. Since $W_t^h(X)$ is a non-negative martingale, to show it is a closed martingale, it suffices to prove

$$\mathbb{P}_\mu \left[W_\infty^h(X) \right] = \langle h, \mu \rangle. \tag{5.20}$$

First we claim that (5.20) is true for $\mu_B(dy) := 1_B(y) \hat{h}(y) m(dy)$ with $0 < m(B) < +\infty$. It is straightforward to see from the change of measure methodology (see, for example, [16, Theorem 5.3.3]) that the proof for this claim is complete as soon as we can show that

$$\mathbb{Q}_{\mu_B} \left(\limsup_{t \rightarrow +\infty} W_t^h(X) < +\infty \right) = 1. \tag{5.21}$$

Since $((X_t)_{t \geq 0}; \mathbb{Q}_{\mu_B})$ is equal in law to $((\Gamma_t)_{t \geq 0}; \mathbb{P}_{\mu_B})$, (5.21) is equivalent to that

$$\mathbb{P}_{\mu_B} \left(\limsup_{t \rightarrow +\infty} W_t^h(\Gamma) < +\infty \right) = 1. \tag{5.22}$$

In the remainder of this proof, we define a function $\log^* \theta := \theta/e$ if $\theta \leq e$ and $\log^* \theta := \log \theta$ if $\theta > e$. Under the assumptions of Theorem 5.1, one can prove by elementary computation that (5.1) implies

$$\left(\int_{(0, +\infty)} r h(\cdot) \log^*(r h(\cdot)) \Pi^L(\cdot, dr), \hat{h} \right) + \left(\int_{(0, +\infty)} r \pi(\cdot, h) \log^*(r \pi(\cdot, h)) \Pi^{NL}(\cdot, dr), 1_A \hat{h} \right) < +\infty. \tag{5.23}$$

Recall that \mathcal{G} is the σ -field generated by $\tilde{\xi}$ (including $\{\tau_i : i \geq 1\}$), $\{D_t^m : t \geq 0\}$, $\{D_t^n : t \geq 0\}$, $\{\Theta_i : i \geq 1\}$ and $\{\Lambda_s^m : s \geq 0\}$. By (4.19), for any $t > 0$,

$$\begin{aligned}
&\mathbb{P}_{\mu_B} \left(W_t^h(\Gamma) \mid \mathcal{G} \right) \\
&= e^{\lambda_1 t} \left(\langle \mathfrak{P}_t h, \mu_B \rangle + \sum_{s \in D_t^n} \mathfrak{P}_{t-s} h(\tilde{\xi}_s) + \sum_{s \in D_t^m} \Lambda_s^m \mathfrak{P}_{t-s} h(\tilde{\xi}_s) + \sum_{\tau_i \leq t} \Theta_i \pi(\tilde{\xi}_{\tau_i-}, \mathfrak{P}_{t-\tau_i} h) \right) \\
&= \langle h, \mu_B \rangle + \sum_{s \in D_t^n} e^{\lambda_1 s} h(\tilde{\xi}_s) + \sum_{s \in D_t^m} e^{\lambda_1 s} \Lambda_s^m h(\tilde{\xi}_s) + \sum_{\tau_i \leq t} e^{\lambda_1 \tau_i} \Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) \\
&\leq \langle h, \mu_B \rangle + \sum_{s \in D_t^n} e^{\lambda_1 s} h(\tilde{\xi}_s) + \sum_{s \in D_t^m} e^{\lambda_1 s} \Lambda_s^m h(\tilde{\xi}_s) + \sum_{i=1}^{+\infty} e^{\lambda_1 \tau_i} \Theta_i \pi(\tilde{\xi}_{\tau_i-}, h). \tag{5.24}
\end{aligned}$$

We begin with the second term on the right hand side of (5.24). Let $\varepsilon \in (0, -\lambda_1)$ be an arbitrary constant.

$$\sum_{s \in D^n} e^{\lambda_1 s} h(\tilde{\xi}_s) = \sum_{s \in D^n} e^{\lambda_1 s} h(\tilde{\xi}_s) 1_{\{h(\tilde{\xi}_s) > e^{\varepsilon s}\}} + \sum_{s \in D^n} e^{\lambda_1 s} h(\tilde{\xi}_s) 1_{\{h(\tilde{\xi}_s) \leq e^{\varepsilon s}\}} =: \text{I} + \text{II}.$$

Recall that given $\tilde{\xi}$, the random measure $\sum_{s \in D^n} \delta_s(\cdot)$ on $[0, +\infty)$ is a Poisson random measure with intensity $2b(\tilde{\xi}_t)dt$, and that $\rho(dx) = h(x)\hat{h}(x)m(dx)$ is an invariant probability measure for \tilde{P}_t . We have by Fubini's theorem,

$$\begin{aligned} \mathbb{P}_{\mu_B}(\text{II}) &= \mathbb{P}_{\mu_B} \left[\mathbb{P}_{\mu_B} \left(\sum_{s \in D^n} e^{\lambda_1 s} h(\tilde{\xi}_s) 1_{\{h(\tilde{\xi}_s) \leq e^{\varepsilon s}\}} \middle| \tilde{\xi}_r : r \geq 0 \right) \right] \\ &= \mathbb{P}_{\mu_B} \left(\int_0^{+\infty} 2b(\tilde{\xi}_s) e^{\lambda_1 s} h(\tilde{\xi}_s) 1_{\{h(\tilde{\xi}_s) \leq e^{\varepsilon s}\}} ds \right) \\ &= \frac{2}{\langle h, \mu_B \rangle} \int_0^{+\infty} e^{\lambda_1 s} ds \int_E \mathbb{P}_{\mu, x} \left[b(\tilde{\xi}_s) h(\tilde{\xi}_s) 1_{\{h(\tilde{\xi}_s) \leq e^{\varepsilon s}\}} \right] 1_B(x) \rho(dx) \\ &\leq \frac{2}{\langle h, \mu_B \rangle} \int_0^{+\infty} e^{\lambda_1 s} ds \int_E b(x) h(x) 1_{\{h(x) \leq e^{\varepsilon s}\}} \rho(dx) \\ &\leq \frac{2\|b\|_\infty}{\langle h, \mu_B \rangle} \int_0^{+\infty} e^{(\lambda_1 + \varepsilon)s} ds \int_E \rho(dx) < +\infty. \end{aligned}$$

Thus we have $\mathbb{P}_{\mu_B}(\text{II} < +\infty) = 1$. On the other hand,

$$\begin{aligned} \mathbb{P}_{\mu_B} \left(\sum_{s \in D^n} 1_{\{h(\tilde{\xi}_s) > e^{\varepsilon s}\}} \right) &= \mathbb{P}_{\mu_B} \left(\int_0^{+\infty} 2b(\tilde{\xi}_s) 1_{\{h(\tilde{\xi}_s) > e^{\varepsilon s}\}} ds \right) \\ &= \frac{2}{\langle h, \mu_B \rangle} \int_0^{+\infty} ds \int_E \mathbb{P}_{\mu, x} \left[b(\tilde{\xi}_s) 1_{\{h(\tilde{\xi}_s) > e^{\varepsilon s}\}} \right] 1_B(x) \rho(dx) \\ &\leq \frac{2}{\langle h, \mu_B \rangle} \int_0^{+\infty} ds \int_E b(x) 1_{\{h(x) > e^{\varepsilon s}\}} \rho(dx) \\ &= \frac{2}{\langle h, \mu_B \rangle} \int_E b(x) \rho(dx) \int_0^{\frac{\log^+ h(x)}{\varepsilon}} ds \\ &\leq \frac{2}{\varepsilon \langle h, \mu_B \rangle} \|b\|_\infty \log^+ \|h\|_\infty < +\infty. \end{aligned}$$

This implies that I is the sum of finitely many terms. Thus we have $\mathbb{P}_{\mu_B}(\text{I} < +\infty) = 1$. For the third term in (5.24), we have

$$\begin{aligned} \sum_{s \in D^m} e^{\lambda_1 s} \Lambda_s^m h(\tilde{\xi}_s) &= \sum_{s \in D^m} e^{\lambda_1 s} \Lambda_s^m h(\tilde{\xi}_s) 1_{\{\Lambda_s^m h(\tilde{\xi}_s) \leq e^{\varepsilon s}\}} + \sum_{s \in D^m} e^{\lambda_1 s} \Lambda_s^m h(\tilde{\xi}_s) 1_{\{\Lambda_s^m h(\tilde{\xi}_s) > e^{\varepsilon s}\}} \\ &=: \text{III} + \text{IV}. \end{aligned}$$

In view of Definition 4.4.(iii), for III, we have

$$\mathbb{P}_{\mu_B}(\text{III}) = \mathbb{P}_{\mu_B} \left(\int_0^{+\infty} \int_E e^{\lambda_1 s} r^2 h(\tilde{\xi}_s) 1_{\{r h(\tilde{\xi}_s) \leq e^{\varepsilon s}\}} \Pi^L(\tilde{\xi}_s, dr) ds \right)$$

$$\begin{aligned}
&= \frac{1}{\langle h, \mu_B \rangle} \int_0^{+\infty} e^{\lambda_1 s} ds \int_E \mathbb{P}_{\mu, x} \left(\int_E r^2 h(\tilde{\xi}_s) 1_{\{rh(\tilde{\xi}_s) \leq e^{\varepsilon s}\}} \Pi^L(\tilde{\xi}_s, dr) \right) 1_B(x) \rho(dx) \\
&\leq \frac{1}{\langle h, \mu_B \rangle} \int_0^{+\infty} e^{\lambda_1 s} ds \int_E \rho(dx) \int_E r^2 h(x) 1_{\{rh(x) \leq e^{\varepsilon s}\}} \Pi^L(x, dr) \\
&= \frac{1}{\langle h, \mu_B \rangle} \int_E \rho(dx) \int_{(0, +\infty)} r^2 h(\cdot) \Pi^L(\cdot, dr) \int_{\log^+ rh(\cdot)/\varepsilon}^{+\infty} e^{\lambda_1 s} ds \\
&= \frac{-1}{\lambda_1 \langle h, \mu_B \rangle} \left(\int_{(0, +\infty)} r^2 h^2(\cdot) (rh(\cdot) \vee 1)^{\lambda_1/\varepsilon} \Pi^L(\cdot, dr), \hat{h} \right) \\
&= \frac{-1}{\lambda_1 \langle h, \mu_B \rangle} \left(\int_{(0, +\infty)} rh(\cdot) \log^*(rh(\cdot)) \left(\frac{rh(\cdot)}{(rh(\cdot) \vee 1)^{-\lambda_1/\varepsilon} \log^*(rh(\cdot))} \right) \Pi^L(\cdot, dr), \hat{h} \right).
\end{aligned} \tag{5.25}$$

Note that the function $r \mapsto \frac{r}{(r \vee 1)^{-\lambda_1/\varepsilon} \log^* r}$ is bounded from above on $(0, +\infty)$. This together with (5.23) implies that the right hand side of (5.25) is finite. It follows that $\mathbb{P}_{\mu_B}(\text{III} < +\infty) = 1$. It has been shown by (5.18) that $\mathbb{P}_{\mu_B} \left(\sum_{s \in D^m} 1_{\{\Lambda_s^m h(\tilde{\xi}_s) > e^{\varepsilon s}\}} < +\infty \right) = 1$. This implies that IV is the sum of finitely many terms. Thus we have $\mathbb{P}_{\mu_B}(\text{IV} < +\infty) = 1$. The fourth term on the right hand side of (5.24) can be dealt with similarly. In fact, we have

$$\begin{aligned}
&\sum_{i=1}^{+\infty} e^{\lambda_1 \tau_i} \Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) \\
&= \sum_{i=1}^{+\infty} e^{\lambda_1 \tau_i} \Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) 1_{\{\Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) \leq e^{\varepsilon \tau_i}\}} + \sum_{i=1}^{+\infty} e^{\lambda_1 \tau_i} \Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) 1_{\{\Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) > e^{\varepsilon \tau_i}\}} \\
&=: \text{V} + \text{VI}.
\end{aligned}$$

Recall that given $\tilde{\xi}$ (including $\{\tau_i : i \geq 1\}$), Θ_i is distributed according to $\eta(\tilde{\xi}_{\tau_i-}, d\theta)$ given by (4.17). Thus by Fubini's theorem and (4.7),

$$\begin{aligned}
&\mathbb{P}_{\mu_B}(\text{V}) \\
&= \mathbb{P}_{\mu_B} \left[\sum_{i=1}^{+\infty} e^{\lambda_1 \tau_i} \pi(\tilde{\xi}_{\tau_i-}, h) \int_{[0, +\infty)} \theta 1_{\{\theta \pi(\tilde{\xi}_{\tau_i-}, h) \leq e^{\varepsilon \tau_i}\}} \eta(\tilde{\xi}_{\tau_i-}, d\theta) \right] \\
&= \mathbb{P}_{\mu_B} \left[\sum_{i=1}^{+\infty} e^{\lambda_1 \tau_i} \frac{\pi(\tilde{\xi}_{\tau_i-}, h)}{\gamma(\tilde{\xi}_{\tau_i-})} 1_A(\tilde{\xi}_{\tau_i-}) \int_{\{0 < \theta \pi(\tilde{\xi}_{\tau_i-}, h) \leq e^{\varepsilon \tau_i}\}} \theta^2 \Pi^{NL}(\tilde{\xi}_{\tau_i-}, d\theta) \right] \\
&= \frac{1}{\langle h, \mu_B \rangle} \int_E \mathbb{P}_{\mu, x} \left[\int_0^{+\infty} q(\tilde{\xi}_s) e^{\lambda_1 s} \frac{\pi(\tilde{\xi}_s, h)}{\gamma(\tilde{\xi}_s)} 1_A(\tilde{\xi}_s) ds \int_{\{0 < \theta \pi(\tilde{\xi}_s, h) \leq e^{\varepsilon s}\}} \theta^2 \Pi^{NL}(\tilde{\xi}_s, d\theta) \right] 1_B(x) \rho(dx) \\
&= \frac{1}{\langle h, \mu_B \rangle} \int_0^{+\infty} e^{\lambda_1 s} ds \int_E \mathbb{P}_{\mu, x} \left[\frac{\pi(\tilde{\xi}_s, h)^2}{h(\tilde{\xi}_s)} 1_A(\tilde{\xi}_s) \int_{\{0 < \theta \pi(\tilde{\xi}_s, h) \leq e^{\varepsilon s}\}} \theta^2 \Pi^{NL}(\tilde{\xi}_s, d\theta) \right] 1_B(x) \rho(dx) \\
&\leq \frac{1}{\langle h, \mu_B \rangle} \int_0^{+\infty} e^{\lambda_1 s} ds \int_E 1_A(x) \pi(x, h)^2 \hat{h}(x) m(dx) \int_{\{0 < \theta \pi(x, h) \leq e^{\varepsilon s}\}} \theta^2 \Pi^{NL}(\tilde{\xi}_s, d\theta) \\
&= \frac{1}{\langle h, \mu_B \rangle} \int_E 1_A(x) \pi(x, h)^2 \hat{h}(x) m(dx) \int_{(0, +\infty)} \theta^2 \Pi^{NL}(\tilde{\xi}_s, d\theta) \int_{\frac{\log^+(\theta \pi(x, h))}{\varepsilon}}^{+\infty} e^{\lambda_1 s} ds
\end{aligned}$$

$$\begin{aligned}
&= \frac{-1}{\lambda_1 \langle h, \mu_B \rangle} \left(\int_{(0, +\infty)} 1_A(\cdot) \pi(\cdot, h)^2 \theta^2 (\pi(\cdot, h) \theta \vee 1)^{\lambda_1/\varepsilon} \Pi^{NL}(\cdot, d\theta), \hat{h} \right) \\
&= \frac{-1}{\lambda_1 \langle h, \mu_B \rangle} \left(\int_{(0, +\infty)} 1_A(\cdot) \pi(\cdot, h) \theta \log^*(\pi(\cdot, h) \theta) \right. \\
&\quad \left. \times \left(\frac{\pi(\cdot, h) \theta}{(\pi(\cdot, h) \theta \vee 1)^{-\lambda_1/\varepsilon} \log^*(\pi(\cdot, h) \theta)} \right) \Pi^{NL}(\cdot, d\theta), \hat{h} \right).
\end{aligned}$$

Since $\theta \mapsto \frac{\theta}{(\theta \vee 1)^{-\lambda_1/\varepsilon} \log^* \theta}$ is bounded from above on $(0, +\infty)$, we get $\mathbb{P}_{\mu_B}(V) < +\infty$ by (5.23), and hence $\mathbb{P}_{\mu_B}(V < +\infty) = 1$. We have shown in (5.19) that $\mathbb{P}_{\mu_B} \left(\sum_{i=1}^{+\infty} 1_{\{\Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) > e^{\varepsilon \tau_i}\}} < +\infty \right) = 1$. Thus VI is the sum of finitely many terms and $\mathbb{P}_{\mu_B}(VI < +\infty) = 1$. The above arguments show that the right hand side of (5.24) is finite almost surely, and hence $\limsup_{t \rightarrow +\infty} \mathbb{P}_{\mu_B}(W_t^h(\Gamma) | \mathcal{G}) < +\infty$ \mathbb{P}_{μ_B} -a.s. By Fatou's lemma, $\mathbb{P}_{\mu_B}(\liminf_{t \rightarrow +\infty} W_t^h(\Gamma) | \mathcal{G}) < +\infty$ \mathbb{P}_{μ_B} -a.s. Let

$$A_n := \left\{ \mathbb{P}_{\mu_B} \left(\liminf_{t \rightarrow +\infty} W_t^h(\Gamma) | \mathcal{G} \right) \leq n \right\} \in \mathcal{G} \quad \text{for } n \geq 1.$$

Then $\mathbb{P}_{\mu_B}(\cup_{n=1}^{+\infty} A_n) = 1$. Since

$$\int_{A_n} \liminf_{t \rightarrow +\infty} W_t^h(\Gamma) d\mathbb{P}_{\mu_B} = \int_{A_n} \mathbb{P}_{\mu_B} \left(\liminf_{t \rightarrow +\infty} W_t^h(\Gamma) | \mathcal{G} \right) d\mathbb{P}_{\mu_B} \leq n,$$

we get $\liminf_{t \rightarrow +\infty} W_t^h(\Gamma) < +\infty$ \mathbb{P}_{μ_B} -a.s on A_n for all $n \geq 1$. Thus

$$\mathbb{P}_{\mu_B} \left(\liminf_{t \rightarrow +\infty} W_t^h(\Gamma) < +\infty \right) = 1.$$

Note that by [23, Proposition 2] $W_t^h(\Gamma)^{-1}$ is a non-negative \mathbb{P}_{μ_B} -supermartingale, which implies that $\lim_{t \rightarrow +\infty} W_t^h(\Gamma)^{-1}$ exists \mathbb{P}_{μ_B} -a.s. It follows that

$$\mathbb{P}_{\mu_B} \left(\limsup_{t \rightarrow +\infty} W_t^h(\Gamma) < +\infty \right) = 1.$$

This proves (5.22) and consequently $\mathbb{P}_{\mu_B}[W_\infty^h(X)] = \langle h, \mu_B \rangle$. Recall that $\mathbb{P}_{\mu_B}[W_\infty^h(X)] = \langle gh, \mu_B \rangle$ where $g(x) = h(x)^{-1} \mathbb{P}_{\delta_x}[W_\infty^h(X)]$. Thus we have

$$\langle gh, \mu_B \rangle = \langle h, \mu_B \rangle. \quad (5.26)$$

Note that $0 \leq g(x) \leq 1$ for every $x \in E$. We get by (5.26) that $g(x) = 1$ m -a.e. on B . Since B is arbitrary, $g(x) = 1$ m -a.e. on E . It then follows from (5.13) that $g(x) = \tilde{P}_t g(x) = \int_E \tilde{p}(t, x, y) g(y) \rho(dy) = 1$ for every $x \in E$. Therefore by (5.12), $\mathbb{P}_\mu[W_\infty^h(X)] = \langle h, \mu \rangle$ holds for all $\mu \in \mathcal{M}(E)$. This completes the proof for Theorem 5.1.(i).

(ii) Suppose $\lambda_1 > 0$. Clearly $\mathbb{P}_\mu(W_\infty^h(X) = 0) = 1$ if and only if $\mathbb{P}_\mu[W_\infty^h(X)] = 0$. By (5.12), this would follow if $g(x) = 0$ for every $x \in E$. Recall that $g(x) = \tilde{P}_t g(x) = \int_E \tilde{p}(t, x, y) g(y) \rho(dy)$. It suffices to prove that $g(x) = 0$ for m -almost every $x \in E$, or equivalently,

$$\mathbb{P}_{\delta_x}[W_\infty^h(X)] = 0 \quad \text{for } m\text{-a.e. } x \in E. \quad (5.27)$$

By the change of measure methodology (see, for example, [16, Theorem 5.3.3]), (5.27) would follow if

$$\mathbb{P}_{\delta_x}(\limsup_{t \rightarrow +\infty} W_t^h(\Gamma) = +\infty) = 1 \quad \text{for } m\text{-a.e. } x \in E. \quad (5.28)$$

By the definition of Γ_t , we have

$$W_s^h(\Gamma) = e^{\lambda_1 s} \langle h, \Gamma_s \rangle \geq e^{\lambda_1 s} \Lambda_s^m h(\tilde{\xi}_s) \quad \text{for } s \in D^m,$$

and

$$W_{\tau_i}^h(\Gamma) \geq e^{\lambda_1 \tau_i} \Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) \quad \text{for } i \geq 1.$$

Thus under \mathbb{P}_{δ_x} ,

$$\begin{aligned} & \limsup_{t \rightarrow +\infty} W_t^h(\Gamma) \\ & \geq \limsup_{D^m \ni s \rightarrow +\infty} e^{\lambda_1 s} \Lambda_s^m h(\tilde{\xi}_s) 1_{\{\Lambda_s^m h(\tilde{\xi}_s) \geq 1\}} \vee \limsup_{i \rightarrow +\infty} e^{\lambda_1 \tau_i} \Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) 1_{\{\Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) \geq 1\}}. \end{aligned} \quad (5.29)$$

Lemma 5.5 implies that for m -a.e. $x \in E$, both $\Lambda_s^m h(\tilde{\xi}_s) 1_{\{\Lambda_s^m h(\tilde{\xi}_s) \geq 1\}}$ and $\Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) 1_{\{\Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) \geq 1\}}$ grow subexponentially. Thus when $\lambda_1 > 0$, the right hand side of (5.29) goes to infinity. Hence we get (5.28) for m -a.e. $x \in E$. □

6 Necessary condition for non-degenerate martingale limit

In this section we will give necessary conditions for the fundamental martingale to have a non-degenerate limit. Recall that $\tilde{p}(t, x, y)$ is the transition density of the spine $\tilde{\xi}$ with respect to the measure ρ defined in (4.4). We start with the following assumption.

Assumption 4.

$$\lim_{t \rightarrow +\infty} \sup_{x \in E} \operatorname{esssup}_{y \in E} |\tilde{p}(t, x, y) - 1| = 0.$$

Proposition 6.1. *Suppose that Assumptions 0-4 hold. Then ρ is an ergodic measure for $(\tilde{P}_t)_{t \geq 0}$ in the sense of [14].*

Proof. Recall that ρ is an invariant probability measure for $(\tilde{P}_t)_{t \geq 0}$. By [14, Theorem 3.2.4], it suffices to prove that for any $\varphi \in L^2(E, \rho)$,

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \tilde{P}_s \varphi ds = \langle \varphi, \rho \rangle \quad \text{in } L^2(E, \rho). \quad (6.1)$$

It follows from Assumption 4 that for any $\varepsilon > 0$, there is $t_0 > 0$ such that

$$\sup_{x \in E} \operatorname{esssup}_{y \in E} |\tilde{p}(s, x, y) - 1| \leq \varepsilon \quad \text{for all } s \geq t_0. \quad (6.2)$$

For $x \in E$ and $t > t_0$,

$$\frac{1}{t} \int_0^t \tilde{P}_s \varphi ds - \langle \varphi, \rho \rangle = \frac{1}{t} \int_0^{t_0} \tilde{P}_s \varphi ds - \frac{t_0}{t} \langle \varphi, \rho \rangle + \frac{1}{t} \int_{t_0}^t ds \int_E (\tilde{p}(s, x, y) - 1) \varphi(y) \rho(dy). \quad (6.3)$$

By (6.2) and Jensen's inequality, we have

$$\begin{aligned} & \left\| \frac{1}{t} \int_{t_0}^t ds \int_E (\tilde{p}(s, x, y) - 1) \varphi(y) \rho(dy) \right\|_{L^2(E, \rho)}^2 \\ &= \frac{1}{t^2} \int_E \rho(dx) \left(\int_{t_0}^t ds \int_E (\tilde{p}(s, x, y) - 1) \varphi(y) \rho(dy) \right)^2 \\ &\leq \frac{t - t_0}{t^2} \int_E \rho(dx) \int_{t_0}^t ds \int_E (\tilde{p}(s, x, y) - 1)^2 \varphi(y)^2 \rho(dy) \\ &\leq \frac{(t - t_0)^2}{t^2} \varepsilon^2 \|\varphi\|_{L^2(E, \rho)}^2. \end{aligned} \quad (6.4)$$

Moreover by Jensen's inequality and (5.11),

$$\begin{aligned} \left\| \frac{1}{t} \int_0^{t_0} \tilde{P}_s \varphi ds \right\|_{L^2(E, \rho)}^2 &= \frac{1}{t^2} \int_E \rho(dx) \left(\int_0^{t_0} \tilde{P}_s \varphi(x) ds \right)^2 \\ &\leq \frac{t_0}{t^2} \int_E \rho(dx) \int_0^{t_0} \tilde{P}_s(\varphi^2)(x) ds \\ &= \frac{t_0^2}{t^2} \int_E \varphi(x)^2 \rho(dx) = \frac{t_0^2}{t^2} \|\varphi\|_{L^2(E, \rho)}^2. \end{aligned} \quad (6.5)$$

By (6.3)–(6.5), we have

$$\begin{aligned} \left\| \frac{1}{t} \int_0^t \tilde{P}_s \varphi ds - \langle \varphi, \rho \rangle \right\|_{L^2(E, \rho)} &\leq \frac{t_0}{t} \|\varphi\|_{L^2(E, \rho)} + \frac{t_0}{t} |\langle \varphi, \rho \rangle| + \frac{t - t_0}{t} \varepsilon \|\varphi\|_{L^2(E, \rho)} \\ &\leq \frac{2t_0}{t} \|\varphi\|_{L^2(E, \rho)} + \frac{t - t_0}{t} \varepsilon \|\varphi\|_{L^2(E, \rho)}. \end{aligned}$$

Letting $t \rightarrow +\infty$ and then $\varepsilon \rightarrow 0$, we get (6.1). \square

Define

$$E_1 := \{x \in E : \text{supp } \Pi^L(x, \cdot) \supseteq [N, +\infty) \text{ for some } N \geq 0\} \quad (6.6)$$

and

$$E_2 := \{x \in A : \text{supp } \Pi^{NL}(x, \cdot) \supseteq [N, +\infty) \text{ for some } N \geq 0\}. \quad (6.7)$$

The main result of this section is the following theorem.

Theorem 6.2. *Suppose that Assumptions 0-4 hold. If one of the following holds:*

(i) $\lambda_1 \geq 0$ and $m(E_1 \cup E_2) > 0$;

(ii) $\left(\int_{(0, +\infty)} rh(\cdot) \log^+(rh(\cdot)) \Pi^L(\cdot, dr), \hat{h} \right) + \left(\int_{(0, +\infty)} r\pi(\cdot, h) \log^+(r\pi(\cdot, h)) \Pi^{NL}(\cdot, dr), 1_A \hat{h} \right) = +\infty$,

then $P_\mu(W_\infty^h(X) = 0) = 1$ for all $\mu \in \mathcal{M}(E)$.

To prove Theorem 6.2, we need the following lemma.

Lemma 6.3. *Suppose that Assumptions 0-4 hold.*

(i) *If $m(E_1 \cup E_2) > 0$, then for m -almost every $x \in E$,*

$$\limsup_{D^m \ni s \rightarrow +\infty} \Lambda_s^m h(\tilde{\xi}_s) \vee \limsup_{i \rightarrow +\infty} \Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) = +\infty \quad \mathbb{P}_{\cdot, x}\text{-a.s.}; \quad (6.8)$$

(ii) *if condition (ii) of Theorem 6.2 holds, then for m -almost every $x \in E$,*

$$\limsup_{D^m \ni s \rightarrow +\infty} \frac{\log^+ \Lambda_s^m h(\tilde{\xi}_s)}{s} \vee \limsup_{i \rightarrow +\infty} \frac{\log^+ \Theta_i \pi(\tilde{\xi}_{\tau_i-}, h)}{\tau_i} = +\infty \quad \mathbb{P}_{\cdot, x}\text{-a.s.} \quad (6.9)$$

Proof. It is easy to see that (6.9) is equivalent to saying that for m -almost every $x \in E$ and all $\lambda < 0$,

$$\limsup_{D^m \ni s \rightarrow +\infty} e^{\lambda s} \Lambda_s^m h(\tilde{\xi}_s) \vee \limsup_{i \rightarrow +\infty} e^{\lambda \tau_i} \Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) = +\infty \quad \mathbb{P}_{\cdot, x}\text{-a.s.}$$

We divide the conditions of this lemma into two cases, and prove the results separately.

Case I: Suppose either one of the following conditions holds:

(I.a) $m(E_1) > 0$;

(I.b) $(\int_{(0, +\infty)} rh(\cdot) \log^+(rh(\cdot)) \Pi^L(\cdot, dr), \hat{h}) = +\infty$.

Let $\lambda < 0$ be an arbitrary constant. To prove (6.8) (resp. (6.9)) under condition (I.a) (resp. (I.b)), it suffices to prove that for m -a.e. $x \in E$ and any $M \geq 1$,

$$\mathbb{P}_{\cdot, x} \left(\sum_{s \in D^m} 1_{\{\Lambda_s^m h(\tilde{\xi}_s) \geq M\}} = +\infty \right) = 1 \quad \left(\text{resp. } \mathbb{P}_{\cdot, x} \left(\sum_{s \in D^m} 1_{\{e^{\lambda s} \Lambda_s^m h(\tilde{\xi}_s) \geq M\}} = +\infty \right) = 1 \right). \quad (6.10)$$

For $0 \leq s \leq t < +\infty$, $\theta \leq 0$ and $M \geq 1$, let $I_\theta(s, t) := \int_s^t dr \int_{(0, +\infty)} u 1_{\{e^{\theta r} u h(\tilde{\xi}_r) \geq M\}} \Pi^L(\tilde{\xi}_r, du)$, and $I_\theta(t) := I_\theta(0, t)$. Recall that, given $\tilde{\xi}$, for any $T > 0$, $\#\{s \in D_T^m : e^{\theta s} \Lambda_s^m h(\tilde{\xi}_s) \geq M\}$ is a Poisson random variable with parameter $I_\theta(T)$. Hence (6.10) would follow if for m -a.e. $x \in E$,

$$\mathbb{P}_{\cdot, x}(I_0(\infty) = +\infty) = 1 \quad (\text{resp. } \mathbb{P}_{\cdot, x}(I_\lambda(\infty) = +\infty) = 1) \quad \text{under condition (I.a) (resp. (I.b))}. \quad (6.11)$$

Let $\nu(dx) := \hat{h}(x)m(dx)$. Clearly $\mathbb{P}_{\cdot, \nu} = \int_E \mathbb{P}_{\cdot, x} \rho(dx)$. Recall that ρ is an invariant measure for \tilde{P}_t . By Fubini's theorem,

$$\begin{aligned} \mathbb{P}_{\cdot, \nu}(I_\theta(T)) &= \int_E \mathbb{P}_{\cdot, x} \left[\int_0^T dr \int_{(0, +\infty)} u 1_{\{e^{\theta r} u h(\tilde{\xi}_r) \geq M\}} \Pi^L(\tilde{\xi}_r, du) \right] \rho(dx) \\ &= \int_0^T dr \int_E \mathbb{P}_{\cdot, x} \left[\int_{(0, +\infty)} u 1_{\{e^{\theta r} u h(\tilde{\xi}_r) \geq M\}} \Pi^L(\tilde{\xi}_r, du) \right] \rho(dx) \\ &= \int_0^T dr \int_E \rho(dx) \int_{(0, +\infty)} u 1_{\{e^{\theta r} u h(x) \geq M\}} \Pi^L(x, du). \end{aligned} \quad (6.12)$$

By the boundedness of h and $x \mapsto \int_{(0,+\infty)} (u \wedge u^2) \Pi^L(x, du)$, we have

$$\begin{aligned}
\mathbb{P}_{\cdot, \nu}(\mathbf{I}_\theta(T)) &\leq T \int_E \widehat{h}(x) m(dx) \int_{u \geq M/h(x)} h(x) u \Pi^L(x, du) \\
&= T \int_E \widehat{h}(x) m(dx) \int_{u \geq M/h(x)} h(x) (1 \vee \frac{1}{u}) (u \wedge u^2) \Pi^L(x, du) \\
&\leq T \left(1 \vee \frac{\|h\|_\infty}{M}\right) \left\| \int_{(0,+\infty)} (u \wedge u^2) \Pi^L(x, du) \right\|_\infty \int_E h(x) \widehat{h}(x) m(dx) < +\infty.
\end{aligned}$$

Thus $\mathbb{P}_{\cdot, \nu}(\mathbf{I}_\theta(T) < +\infty) = 1$. On the other hand, by the Markov property of $\widetilde{\xi}$ and (6.12),

$$\begin{aligned}
&\mathbb{P}_{\cdot, \nu}(\mathbf{I}_\theta(T)^2) \\
&= \int_E \mathbb{P}_{\cdot, x}(\mathbf{I}_\theta(T)^2) \rho(dx) \\
&= 2 \int_E \rho(dx) \mathbb{P}_{\cdot, x} \left[\int_0^T ds \int_{(0,+\infty)} u 1_{\{e^{\theta s} u h(\widetilde{\xi}_s) \geq M\}} \Pi^L(\widetilde{\xi}_s, du) \right. \\
&\quad \left. \times \int_s^T dr \int_{(0,+\infty)} v 1_{\{e^{\theta r} v h(\widetilde{\xi}_r) \geq M\}} \Pi^L(\widetilde{\xi}_r, dv) \right] \\
&= 2 \int_E \rho(dx) \mathbb{P}_{\cdot, x} \left[\int_0^T ds \int_{(0,+\infty)} u 1_{\{e^{\theta s} u h(\widetilde{\xi}_s) \geq M\}} \Pi^L(\widetilde{\xi}_s, du) \right. \\
&\quad \left. \times \mathbb{P}_{\cdot, \widetilde{\xi}_s} \left(\int_0^{T-s} dr \int_{(0,+\infty)} v 1_{\{e^{\theta(r+s)} v h(\widetilde{\xi}_r) \geq M\}} \Pi^L(\widetilde{\xi}_r, dv) \right) \right] \\
&= 2 \int_E \rho(dx) \int_0^T ds \int_{(0,+\infty)} u 1_{\{e^{\theta s} u h(x) \geq M\}} \Pi^L(x, du) \\
&\quad \times \mathbb{P}_{\cdot, x} \left(\int_0^{T-s} dr \int_{(0,+\infty)} v 1_{\{e^{\theta(r+s)} v h(\widetilde{\xi}_r) \geq M\}} \Pi^L(\widetilde{\xi}_r, dv) \right) \\
&\leq 2 \int_E \rho(dx) \int_0^T ds \int_{(0,+\infty)} u 1_{\{e^{\theta s} u h(x) \geq M\}} \Pi^L(x, du) \\
&\quad \times \mathbb{P}_{\cdot, x} \left(\int_0^T dr \int_{(0,+\infty)} v 1_{\{e^{\theta r} v h(\widetilde{\xi}_r) \geq M\}} \Pi^L(\widetilde{\xi}_r, dv) \right) \\
&= 2 \int_E \rho(dx) \int_0^T ds \int_{(0,+\infty)} u 1_{\{e^{\theta s} u h(x) \geq M\}} \Pi^L(x, du) \mathbb{P}_{\cdot, x}(\mathbf{I}_\theta(T)). \tag{6.13}
\end{aligned}$$

Assumption 4 implies that there are constants $t_1, \delta > 0$ such that

$$\sup_{x \in E} \text{esssup}_{y \in E} \widetilde{p}(t, x, y) \leq 1 + \delta \quad \text{for all } t \geq t_1. \tag{6.14}$$

Using Fubini's theorem, (6.14) and (6.12), we have for $T > t_1$,

$$\begin{aligned}
\mathbb{P}_{\cdot, x}[\mathbf{I}_\theta(t_1, T)] &= \int_{t_1}^T dr \int_E \widetilde{p}(r, x, y) \rho(dy) \int_{(0,+\infty)} v 1_{\{e^{\theta r} v h(y) \geq M\}} \Pi^L(y, dv) \\
&\leq (1 + \delta) \int_{t_1}^T dr \int_E \rho(dy) \int_{(0,+\infty)} v 1_{\{e^{\theta r} v h(y) \geq M\}} \Pi^L(y, dv)
\end{aligned}$$

$$\leq (1 + \delta) \mathbb{P}_{\cdot, \nu}(\mathbf{I}_\theta(T)). \quad (6.15)$$

For $x \in E$,

$$\begin{aligned} \mathbb{P}_{\cdot, x}(\mathbf{I}_\theta(t_1)) &= \mathbb{P}_{\cdot, x} \left[\int_0^{t_1} dr \int_{(0, +\infty)} v 1_{\{e^{\theta r} v h(\tilde{\xi}_r) \geq M\}} \Pi^L(\tilde{\xi}_r, dv) \right] \\ &= \int_0^{t_1} dr \int_E \tilde{p}(r, x, y) \rho(dy) \int_{(0, +\infty)} v 1_{\{e^{\theta r} v h(y) \geq M\}} \Pi^L(y, dv) \\ &\leq \int_0^{t_1} dr \int_E \tilde{p}(r, x, y) \rho(dy) \int_{v \geq M/h(y)} \left(1 \vee \frac{1}{v}\right) (v \wedge v^2) \Pi^L(y, dv) \\ &\leq t_1 \left(1 \vee \frac{\|h\|_\infty}{M}\right) \left\| \int_{(0, +\infty)} (v \wedge v^2) \Pi^L(\cdot, dv) \right\|_\infty =: c_1 < +\infty. \end{aligned} \quad (6.16)$$

It follows from (6.15) and (6.16) that for $T > t_1$,

$$\mathbb{P}_{\cdot, x}(\mathbf{I}_\theta(T)) = \mathbb{P}_{\cdot, x}(\mathbf{I}_\theta(t_1)) + \mathbb{P}_{\cdot, x}(\mathbf{I}_\theta(t_1, T)) \leq c_1 + (1 + \delta) \mathbb{P}_{\cdot, \nu}(\mathbf{I}_\theta(T)).$$

This together with (6.12) and (6.13) implies that

$$\mathbb{P}_{\cdot, \nu}(\mathbf{I}_\theta(T)^2) \leq 2c_1 \mathbb{P}_{\cdot, \nu}(\mathbf{I}_\theta(T)) + 2(1 + \delta) \mathbb{P}_{\cdot, \nu}(\mathbf{I}_\theta(T))^2.$$

Hence by Cauchy-Schwarz inequality, we have

$$\mathbb{P}_{\cdot, \nu} \left(\mathbf{I}_\theta(T) \geq \frac{1}{2} \mathbb{P}_{\cdot, \nu}(\mathbf{I}_\theta(T)) \right) \geq \frac{\mathbb{P}_{\cdot, \nu}(\mathbf{I}_\theta(T))^2}{4 \mathbb{P}_{\cdot, \nu}(\mathbf{I}_\theta(T)^2)} \geq \frac{\mathbb{P}_{\cdot, \nu}(\mathbf{I}_\theta(T))}{8c_1 + 8(1 + \delta) \mathbb{P}_{\cdot, \nu}(\mathbf{I}_\theta(T))}. \quad (6.17)$$

Recall that $\mathbb{P}_{\cdot, \nu}(\mathbf{I}_0(T)) = T \int_E \rho(dx) \int_{u \geq h(x)/M} u \Pi^L(x, du)$. Condition (I.a) implies that the integral on the right hand side is positive. Hence $\mathbb{P}_{\cdot, \nu}(\mathbf{I}_0(T)) \rightarrow +\infty$ as $T \rightarrow +\infty$. On the other hand, note that by (6.12) and Fubini's theorem, for $\lambda < 0$,

$$\begin{aligned} \mathbb{P}_{\cdot, \nu}(\mathbf{I}_\lambda(T)) &\leq \int_E \hat{h}(x) m(dx) \int_{(0, +\infty)} h(x) u \Pi^L(x, du) \int_0^T 1_{\{s \leq \frac{\log^+(h(x)u) - \log M}{-\lambda}\}} ds \\ &= \int_E \hat{h}(x) m(dx) \int_{(0, +\infty)} h(x) u \left(T \wedge \frac{\log^+(h(x)u) - \log M}{-\lambda} \right)^+ \Pi^L(x, du). \end{aligned}$$

Clearly condition (I.b) implies that $\lim_{T \rightarrow +\infty} \mathbb{P}_{\cdot, \nu}(\mathbf{I}_\lambda(T)) = +\infty$. Thus by letting $T \rightarrow +\infty$ in (6.17), we get $\mathbb{P}_{\cdot, \nu}(\mathbf{I}_0(\infty) = +\infty) > 0$ (resp. $\mathbb{P}_{\cdot, \nu}(\mathbf{I}_\lambda(\infty) = +\infty) > 0$) under condition (I.a) (resp. (I.b)). Since $\{\mathbf{I}_0(\infty) = +\infty\}$ (resp. $\{\mathbf{I}_\lambda(\infty) = +\infty\}$) is an invariant event of the canonical dynamic system associated with $(\tilde{P}_t)_{t \geq 0}$ and ergodic measure ρ , it follows from [14, Theorem 1.2.4] that $\mathbb{P}_{\cdot, \nu}(\mathbf{I}_0(\infty) = +\infty) = 1$ (resp. $\mathbb{P}_{\cdot, \nu}(\mathbf{I}_\lambda(\infty) = +\infty) = 1$) under condition (I.a) (resp. (I.b)). Hence we prove (6.11).

Case II. Suppose either one of the following conditions holds:

$$(II.a) \quad m(E_2) > 0;$$

$$(II.b) \quad (1_A(\cdot) \int_{(0, +\infty)} \pi(\cdot, h) r \log^+(\pi(\cdot, h) r) \Pi^{NL}(\cdot, dr), \hat{h}) = +\infty.$$

Let $\lambda < 0$ be an arbitrary constant. To prove (6.8) (resp. (6.9)) under condition (II.a) (resp. (II.b)), it suffices to prove that for m -a.e. $x \in E$ and any $M \geq 1$,

$$\mathbb{P}_{\cdot, x} \left(\sum_{i=1}^{+\infty} 1_{\{\Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) \geq M\}} = +\infty \right) = 1 \quad \left(\text{resp. } \mathbb{P}_{\cdot, x} \left(\sum_{i=1}^{+\infty} 1_{\{e^{\lambda \tau_i} \Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) \geq M\}} = +\infty \right) = 1 \right). \quad (6.18)$$

The main idea of this proof is similar to that of Case I. For any $T > 0$, $\theta \leq 0$ and $M \geq 1$, let $\Pi_\theta(T) := \sum_{\tau_i \leq T} 1_{\{e^{\theta \tau_i} \Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) \geq M\}}$. For any $s \geq 0$ and $x \in E$, define

$$f_\theta(s, x) := \int_{[0, +\infty)} 1_{\{e^{\theta s} u \pi(x, h) \geq M\}} \eta(x, du) = \frac{1}{\gamma(x)} 1_A(x) \int_{(0, +\infty)} 1_{\{e^{\theta s} u \pi(x, h) \geq M\}} u \Pi^{NL}(x, du),$$

$$g_\theta(s, x) := q(x) f_\theta(s, x) = \frac{\pi(x, h)}{h(x)} 1_A(x) \int_{(0, +\infty)} 1_{\{e^{\theta s} u \pi(x, h) \geq M\}} u \Pi^{NL}(x, du).$$

Recall that, given $\tilde{\xi}$ (including $\{\tau_i : i \geq 1\}$), Θ_i is distributed according to $\eta(\tilde{\xi}_{\tau_i-}, dr)$. By (4.7), we have for $x \in E$,

$$\begin{aligned} \mathbb{P}_{\cdot, x}(\Pi_\theta(T)) &= \mathbb{P}_{\cdot, x} \left[\sum_{\tau_i \leq T} \int_{[0, +\infty)} 1_{\{e^{\theta \tau_i} r \pi(\tilde{\xi}_{\tau_i-}, h) \geq M\}} \eta(\tilde{\xi}_{\tau_i-}, dr) \right] \\ &= \mathbb{P}_{\cdot, x} \left[\sum_{\tau_i \leq T} f_\theta(\tau_i, \tilde{\xi}_{\tau_i-}) \right] \\ &= \mathbb{P}_{\cdot, x} \left[\int_0^T q(\tilde{\xi}_s) f_\theta(s, \tilde{\xi}_s) ds \right] = \mathbb{P}_{\cdot, x} \left[\int_0^T g_\theta(s, \tilde{\xi}_s) ds \right]. \end{aligned} \quad (6.19)$$

We still use ν to denote the measure $\hat{h}(x)m(dx)$. Since ρ is an invariant measure for \tilde{P}_t , by Fubini's theorem,

$$\begin{aligned} &\mathbb{P}_{\cdot, \nu}(\Pi_\theta(T)) \\ &= \int_E \mathbb{P}_{\cdot, x}(\Pi_\theta(T)) \rho(dx) \\ &= \int_0^T ds \int_E \mathbb{P}_{\cdot, x}(g_\theta(s, \tilde{\xi}_s)) \rho(dx) \\ &= \int_0^T ds \int_E g_\theta(s, x) \rho(dx) \end{aligned} \quad (6.20)$$

$$= \int_E 1_A(x) \hat{h}(x) m(dx) \int_{(0, +\infty)} \pi(x, h) r \Pi^{NL}(x, dr) \int_0^T 1_{\{e^{\theta s} r \pi(x, h) \geq M\}} ds. \quad (6.21)$$

It then follows by Assumption 3.(ii) that

$$\text{RHS of (6.21)} \leq T \left\| \int_{(0, +\infty)} y \Pi^{NL}(\cdot, dy) \right\|_\infty \int_A \pi(x, h) \hat{h}(x) m(dx) < +\infty.$$

Therefore $\mathbb{P}_{\cdot, \nu}(\Pi_\theta(T) < +\infty) = 1$. Recall that given $\tilde{\xi}$ (including $\{\tau_i : i \geq 1\}$), $\{\Theta_i : i \geq 1\}$ are mutually independent, we have

$$\mathbb{P}_{\cdot, x}(\Pi_\theta(T)^2) - \mathbb{P}_{\cdot, x}(\Pi_\theta(T))$$

$$\begin{aligned}
&= \mathbb{P}_{\cdot, x} \left[\sum_{\tau_i, \tau_j \leq T, i \neq j} 1_{\{e^{\theta \tau_i} \Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) \geq M\}} 1_{\{e^{\theta \tau_j} \Theta_j \pi(\tilde{\xi}_{\tau_j-}, h) \geq M\}} \right] \\
&= \mathbb{P}_{\cdot, x} \left[\sum_{\tau_i, \tau_j \leq T, i \neq j} \int_{[0, +\infty)} 1_{\{e^{\theta \tau_i} y \pi(\tilde{\xi}_{\tau_i-}, h) \geq M\}} \eta(\tilde{\xi}_{\tau_i-}, dy) \int_{[0, +\infty)} 1_{\{e^{\theta \tau_j} z \pi(\tilde{\xi}_{\tau_j-}, h) \geq M\}} \eta(\tilde{\xi}_{\tau_j-}, dz) \right] \\
&= \mathbb{P}_{\cdot, x} \left[\sum_{\tau_i, \tau_j \leq T, i \neq j} f_\theta(\tau_i, \tilde{\xi}_{\tau_i-}) f_\theta(\tau_j, \tilde{\xi}_{\tau_j-}) \right].
\end{aligned}$$

Thus by (4.8),

$$\begin{aligned}
&\mathbb{P}_{\cdot, x} (\Pi_\theta(T)^2) - \mathbb{P}_{\cdot, x} (\Pi_\theta(T)) \\
&= 2\mathbb{P}_{\cdot, x} \left[\int_0^T q(\tilde{\xi}_s) f_\theta(s, \tilde{\xi}_s) ds \int_E \tilde{\Pi}_y \left(\int_0^{T-s} q(\tilde{\xi}_r) f_\theta(s+r, \tilde{\xi}_r) dr \right) \pi^h(\tilde{\xi}_s, dy) \right] \\
&= 2\mathbb{P}_{\cdot, x} \left[\int_0^T g_\theta(s, \tilde{\xi}_s) ds \int_E \tilde{\Pi}_y \left(\int_0^{T-s} g_\theta(s+r, \tilde{\xi}_r) dr \right) \pi^h(\tilde{\xi}_s, dy) \right]. \tag{6.22}
\end{aligned}$$

Note that for each $x \in E$ and $\theta \leq 0$, $s \mapsto g_\theta(s, x)$ is non-increasing. Thus it follows from (6.22) that

$$\mathbb{P}_{\cdot, x} (\Pi_\theta(T)^2) \leq \mathbb{P}_{\cdot, x} (\Pi_\theta(T)) + 2\mathbb{P}_{\cdot, x} \left[\int_0^T g_\theta(s, \tilde{\xi}_s) ds \int_E \tilde{\Pi}_y \left(\int_0^T g_\theta(r, \tilde{\xi}_r) dr \right) \pi^h(\tilde{\xi}_s, dy) \right]. \tag{6.23}$$

By Fubini's theorem, (6.14) and (6.20), we have for $y \in E$ and $T > t_1$,

$$\begin{aligned}
\tilde{\Pi}_y \left(\int_{t_1}^T g_\theta(s, \tilde{\xi}_s) ds \right) &= \int_{t_1}^T ds \int_E \tilde{p}(s, y, z) g_\theta(s, z) \rho(dz) \\
&\leq (1 + \delta) \int_{t_1}^T ds \int_E g_\theta(s, z) \rho(dz) \\
&\leq (1 + \delta) \mathbb{P}_{\cdot, \nu} (\Pi_\theta(T)). \tag{6.24}
\end{aligned}$$

On the other hand, by Assumption 3.(iii),

$$\begin{aligned}
&\sup_{y \in E} \tilde{\Pi}_y \left(\int_0^{t_1} g_\theta(s, \tilde{\xi}_s) ds \right) \\
&= \sup_{y \in E} \int_0^{t_1} ds \int_E \tilde{p}(s, y, z) \frac{\pi(z, h)}{h(z)} 1_A(z) \rho(dz) \int_{(0, +\infty)} r 1_{\{e^{\theta s} r \pi(z, h) \geq M\}} \Pi^{NL}(z, dr) \\
&\leq \left\| \frac{\pi(\cdot, h)}{h} 1_A \right\|_\infty \left\| \int_{(0, +\infty)} r \Pi^{NL}(\cdot, dr) \right\|_\infty \sup_{y \in E} \int_0^{t_1} ds \int_A \tilde{p}(s, y, z) \rho(dz) \\
&\leq c_2 t_1 =: c_3 < +\infty.
\end{aligned}$$

This and (6.24) imply that

$$\tilde{\Pi}_y \left(\int_0^T g_\theta(s, \tilde{\xi}_s) ds \right) \leq c_3 + (1 + \delta) \mathbb{P}_{\cdot, \nu} (\Pi_\theta(T)) \quad \text{for all } y \in E \text{ and } T > t_1.$$

This together with (6.23) and (6.19) implies that

$$\mathbb{P}_{\cdot, x}(\Pi_\theta(T)^2) \leq (1 + 2c_3)\mathbb{P}_{\cdot, x}(\Pi_\theta(T)) + 2(1 + \delta)\mathbb{P}_{\cdot, \nu}(\Pi_\theta(T))\mathbb{P}_{\cdot, x}(\Pi_\theta(T)).$$

Consequently,

$$\mathbb{P}_{\cdot, \nu}(\Pi_\theta(T)^2) = \int_E \mathbb{P}_{\cdot, x}(\Pi_\theta(T)^2) \rho(dx) \leq (1 + 2c_3)\mathbb{P}_{\cdot, \nu}(\Pi_\theta(T)) + 2(1 + \delta)\mathbb{P}_{\cdot, \nu}(\Pi_\theta(T))^2.$$

Recall that $\mathbb{P}_{\cdot, \nu}(\Pi_0(T)) = T \int_A \pi(x, h) \hat{h}(x) m(dx) \int_{r \geq M/\pi(x, h)} r \Pi^{NL}(x, dr)$. Condition (II.a) implies that the integral on the right hand side is positive. Thus $\mathbb{P}_{\cdot, \nu}(\Pi_0(T)) \rightarrow +\infty$ as $T \rightarrow +\infty$. On the other hand, note that by (6.21) and Fubini's theorem, for $\lambda < 0$,

$$\mathbb{P}_{\cdot, \nu}(\Pi_\lambda(T)) = \int_E 1_A(x) \hat{h}(x) m(dx) \int_{(0, +\infty)} \pi(x, h) r \left(\frac{\log^+(\pi(x, h)r) - \log M}{-\lambda} \wedge T \right)^+ \Pi^{NL}(x, dr).$$

Clearly condition (II.b) implies that $\lim_{T \rightarrow +\infty} \mathbb{P}_{\cdot, \nu}(\Pi_\lambda(T)) = +\infty$. Similarly by using Cauchy-Schwarz inequality and letting $T \rightarrow +\infty$, we get $\mathbb{P}_{\cdot, \nu}(\Pi_0(\infty) = +\infty) > 0$ (resp. $\mathbb{P}_{\cdot, \nu}(\Pi_\lambda(\infty) = +\infty) > 0$) under condition (II.a) (resp. (II.b)).

For each $n \geq 1$, we denote by $\tilde{\mathcal{G}}_n$ the σ -field generated by $\tilde{\xi}$ up to time τ_n (including $\{\tau_1, \dots, \tau_n\}$) and $\{\Theta_i : i \leq n\}$. Obviously for each $i \geq 1$, both

$$1_{\{e^{\theta\tau_i} \Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) \geq M\}} \quad \text{and} \quad \int_{[0, +\infty)} 1_{\{e^{\theta\tau_i} r \pi(\tilde{\xi}_{\tau_i-}, h) \geq M\}} \eta(\tilde{\xi}_{\tau_i-}, dr)$$

are $\tilde{\mathcal{G}}_i$ -measurable. Moreover for every $x \in E$, under $\mathbb{P}_{\cdot, x}$,

$$\mathbb{P}_{\cdot, x} \left(1_{\{e^{\theta\tau_{i+1}} \Theta_{i+1} \pi(\tilde{\xi}_{\tau_{i+1}-}, h) \geq M\}} \mid \tilde{\mathcal{G}}_i \right) = \mathbb{P}_{\cdot, x} \left(\int_{[0, +\infty)} 1_{\{e^{\theta\tau_{i+1}} r \pi(\tilde{\xi}_{\tau_{i+1}-}, h) \geq M\}} \eta(\tilde{\xi}_{\tau_{i+1}-}, dr) \mid \tilde{\mathcal{G}}_i \right)$$

Applying the second Borel-Cantelli lemma (see, for example, [16, Corollary 5.3.2]) to both sides of the above equality, we get that

$$\left\{ \sum_{i=1}^{+\infty} 1_{\{e^{\theta\tau_i} \Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) \geq M\}} = +\infty \right\} = \left\{ \sum_{i=1}^{+\infty} \int_{[0, +\infty)} 1_{\{e^{\theta\tau_i} r \pi(\tilde{\xi}_{\tau_i-}, h) \geq M\}} \eta(\tilde{\xi}_{\tau_i-}, dr) = +\infty \right\}$$

under $\mathbb{P}_{\cdot, x}$. It is easy to see from the above representation that $\{\Pi_0(\infty) = +\infty\}$ (resp. $\{\Pi_\lambda(\infty) = +\infty\}$) is an invariant event of the canonical dynamic system associated with $(\tilde{P}_t)_{t \geq 0}$ and ergodic measure ρ , so it follows from [14, Theorem 1.2.4] that $\mathbb{P}_{\cdot, \nu}(\Pi_0(\infty) = +\infty) = 1$ (resp. $\mathbb{P}_{\cdot, \nu}(\Pi_\lambda(\infty) = +\infty) = 1$) under condition (II.a) (resp. (II.b)). Thus (6.18) is valid. \square

Proof of Theorem 6.2. Applying the same argument as in the beginning of the proof of Theorem 5.1.(ii) here, we only need to show that under the assumptions of Theorem 6.2,

$$\mathbb{P}_{\delta_x}(\limsup_{t \rightarrow +\infty} W_t^h(\Gamma) = +\infty) = 1 \quad \text{for } m\text{-a.e. } x \in E.$$

In view of (5.29), this would follow if for m -a.e. $x \in E$,

$$\limsup_{D^m \ni s \rightarrow +\infty} e^{\lambda_1 s} \Lambda_s^m h(\tilde{\xi}_s) \vee \limsup_{i \rightarrow +\infty} e^{\lambda_1 \tau_i} \Theta_i \pi(\tilde{\xi}_{\tau_i-}, h) = +\infty \quad \mathbb{P}_{\cdot, x}\text{-a.s.}$$

which, under the assumptions of this theorem, is automatically true by Lemma 6.3. Hence we complete the proof. \square

The following corollaries follow directly from Theorem 5.1 and Theorem 6.2.

Corollary 6.4. *Suppose that Assumptions 0-4 hold and that $m(E_1 \cup E_2) > 0$ with E_1 and E_2 defined in (6.6) and (6.7) respectively. For every $\mu \in \mathcal{M}(E)^0$, $W_\infty^h(X)$ is non-degenerate if and only if $\lambda_1 < 0$ and condition (5.1) holds. Moreover, X_t under P_μ exhibits weak local extinction if $\lambda_1 \geq 0$.*

Corollary 6.5. *Suppose Assumptions 0-4 hold and $\lambda_1 < 0$. For every $\mu \in \mathcal{M}(E)^0$, $W_\infty^h(X)$ is non-degenerate if and only if condition (5.1) holds.*

Remark 6.6. (i) Suppose that $\{Z_n : n \geq 1\}$ is a Galton-Watson branching process with each particle having probability p_n of giving birth to n children. Let L stand for a random variable with this offspring distribution. Let $m := \sum_{n=0}^{+\infty} np_n$ be the mean number of offspring per particle. Then Z_n/m^n is a non-negative martingale. Kesten and Stigum proved that when $1 < m < +\infty$, the limit of Z_n/m^n is non-degenerate if and only if $E(L \log^+ L) < +\infty$. This result is usually referred to the Kesten-Stigum $L \log L$ theorem. Corollary 6.5 can be viewed as a natural analogue of the Kesten-Stigum $L \log L$ theorem for superprocesses.

(ii) Note that in the case of purely local branching mechanism, Assumption 4 can be written as

$$\lim_{t \rightarrow +\infty} \sup_{x \in E} \operatorname{esssup}_{y \in E} |p^h(t, x, y) - 1| = 0,$$

where $p^h(t, x, y)$ denotes the transition density function of ξ^h with respect to the measure ρ . If E is a bounded domain in \mathbb{R}^d , m is the Lebesgue measure on \mathbb{R}^d and ξ is a symmetric diffusion on E , then $a(x) \in \mathcal{B}_b(E) \subset \mathbf{K}(\xi) \cap L^2(E, m)$. Hence for the class of superdiffusions with local branching mechanisms considered in [32], our Assumptions 0-4 hold and Corollary 6.5 generalizes [32, Theorem 1.1].

7 Examples

In this section, we will give examples satisfying our assumptions. We will not try to give the most general examples possible.

Example 7.1. Suppose $E = \{1, 2, \dots, K\}$ ($K \geq 2$), m is the counting measure on E and $P_t f(i) = f(i)$ for all $i \in E$, $t \geq 0$ and $f \in \mathcal{B}^+(E)$. Suppose

$$\phi^L(i, \lambda) := a(i)\lambda + b(i)\lambda^2 + \int_{(0, +\infty)} (e^{-\lambda r} - 1 + \lambda r) \Pi^L(i, dr),$$

$$\phi^{NL}(i, f) := -c(i)\pi(i, f) - \int_{(0, +\infty)} (1 - e^{-r\pi(i, f)}) \Pi^{NL}(i, dr),$$

where for each $i \in E$, $a(i) \in (-\infty, +\infty)$, $b(i), c(i) \geq 0$, $(r \wedge r^2)\Pi^L(i, dr)$ and $r\Pi^{NL}(i, dr)$ are bounded kernels from E to $(0, +\infty)$ with $\{i \in E : \int_{(0, +\infty)} r\Pi^{NL}(i, dr) > 0\} \neq \emptyset$, and $\pi(i, dj)$ is a

probability kernel on E with $\pi(i, \{i\}) = 0$ for every $i \in E$. As a special case of the model given in Section 2.1, we have a non-local branching superprocess $\{X_t : t \geq 0\}$ in $\mathcal{M}(E)$ with transition probabilities given by

$$P_\mu [\exp(-\langle f, X_t \rangle)] = \exp(-\langle V_t f, \mu \rangle) \quad \text{for } \mu \in \mathcal{M}(E), \ t \geq 0 \text{ and } f \in \mathcal{B}_b^+(E),$$

where $V_t f(i)$ is the unique non-negative locally bounded solution to the following integral equation:

$$V_t f(i) = f(i) - \int_0^t (\phi^L(i, V_s f(i)) + \phi^{NL}(i, V_s f)) ds \quad \text{for } t \geq 0, \ i \in E.$$

For every $i \in E$ and $\mu \in \mathcal{M}(E)$, we define $\mu^{(i)} := \mu(\{i\})$. The map $\mu \mapsto (\mu^{(1)}, \dots, \mu^{(K)})^T$ is clearly a homeomorphism between $\mathcal{M}(E)$ and the K -dimensional product space $[0, +\infty)^K$. Hence $\{(X_t^{(1)}, \dots, X_t^{(K)})^T : t \geq 0\}$ is a Markov process in $[0, +\infty)^K$, which is called a K -type continuous-state branching process. (Clearly the 1-type continuous-state branching process defined in a similar way coincides with the classical one-dimensional continuous-state branching process, see, for example, [31, Chapter 3].) For simplicity, we assume $b(i) \equiv 0$. For $i, j \in E$, let $p_{ij} := \pi(i, \{j\})$ and $\gamma(i) := c(i) + \int_{(0, +\infty)} r \Pi^{NL}(i, dr)$. Define the $K \times K$ matrix $M(t) = (M(t)_{ij})_{ij}$ by $M(t)_{ij} := P_{\delta_i} [X_t^{(j)}]$ for $i, j \in E$. Let \mathfrak{P}_t denote the mean semigroup of X , that is

$$\mathfrak{P}_t f(i) := P_{\delta_i} [\langle f, X_t \rangle] = \sum_{j=1}^K M(t)_{ij} f(j) \quad \text{for } i \in E, \ t \geq 0 \text{ and } f \in \mathcal{B}^+(E).$$

By the Markov property and (2.6), $M(t)$ satisfies that

$$\begin{aligned} M(0) &= I, \quad M(t+s) = M(t)M(s) \quad \text{for } t, s \geq 0, \\ \text{and} \quad M(t)_{ij} &= \delta_j(i) - a(i) \int_0^t M(s)_{ij} ds + \gamma(i) \sum_{k=1}^K p_{ik} \int_0^t M(s)_{kj} ds, \quad \text{for } i, j \in E. \end{aligned} \quad (7.1)$$

This implies that $M(t)$ has a formal matrix generator $A := (A_{ij})_{ij}$ given by

$$M(t) = e^{At} = \sum_{n=0}^{+\infty} t^n \frac{A^n}{n!}, \quad \text{and } A_{ij} = \gamma(i)p_{ij} - a(i)\delta_i(j) \text{ for } i, j \in E.$$

We assume A is an irreducible matrix. It then follows by [3, Lemma A.1] that $M(t)_{ij} > 0$ for all $t > 0$ and $i, j \in E$. Let $\Lambda := \sup_{\lambda \in \sigma(A)} \operatorname{Re}(\lambda)$ where $\sigma(A)$ denotes the set of eigenvalues of A . The Perron-Frobenius theory (see, for example, [3, Lemma A.3]) tells us that for every $t > 0$, $e^{\Lambda t}$ is a simple eigenvalue of $M(t)$, and there exist a unique positive right eigenvector $\mathbf{u} = (u_1, \dots, u_K)^T$ and a unique positive left eigenvector $\mathbf{v} = (v_1, \dots, v_K)^T$ such that

$$\sum_{i=1}^K u_i = \sum_{i=1}^K u_i v_i = 1, \quad M(t)\mathbf{u} = e^{\Lambda t}\mathbf{u}, \quad \mathbf{v}^T M(t) = e^{\Lambda t}\mathbf{v}.$$

Moreover it is known by [3, Lemma A.3] that for each $i, j \in E$,

$$e^{-\Lambda t} M(t)_{ij} \rightarrow u_i v_j \quad \text{as } t \rightarrow +\infty. \quad (7.2)$$

One can easily verify that Assumptions 0-3 hold with $\lambda_1 = -\Lambda$, $h(i) = cu_i$ and $\hat{h}(i) = c^{-1}v_i$, where $c := \left(\sum_{j=1}^K u_j^2\right)^{-1/2}$ is a positive constant. Thus $W_t^h(X) := ce^{-\Lambda t} \sum_{i=1}^K u_i X_t^{(i)}$ is a non-negative martingale. Applying Theorem 4.6 here, we can deduce that under the martingale change of measure the spine process $\tilde{\xi}$ is a continuous-time Markov process on E with Q -matrix $Q = (q_{ij})_{ij}$ given by

$$q_{ii} := -\frac{\gamma(i) \sum_{j=1}^K p_{ij} u_j}{u_i} = -(\Lambda + a(i)), \quad q_{ij} := \frac{\gamma(i) p_{ij} u_j}{u_i} \quad \text{for } i \neq j.$$

Let $\rho(dj) := u_j v_j m(dj) = \sum_{i=1}^K u_j v_j \delta_i(dj)$. Let \tilde{P}_t denote the transition semigroup of the spine $\tilde{\xi}$ and $\tilde{p}(t, i, j)$ denote its transition density with respect to ρ . It follows by Proposition 4.1 that for each $i, j \in E$,

$$\tilde{p}(t, i, j) u_j v_j = \int_E \tilde{p}(t, i, k) \delta_j(k) \rho(dk) = \tilde{P}_t \delta_j(i) = \frac{e^{-\Lambda t}}{h(i)} \mathfrak{P}_t(h \delta_j)(i) = \frac{e^{-\Lambda t}}{u_i} M(t)_{ij} u_j.$$

Thus $\tilde{p}(t, i, j) = e^{-\Lambda t} (u_i v_j)^{-1} M(t)_{ij}$. By (7.2), we have for each $i, j \in E$.

$$\tilde{p}(t, i, j) \rightarrow 1 \quad \text{as } t \rightarrow +\infty.$$

Hence Assumption 4 also holds for this example. Applying Corollary 6.4 here, we conclude that for every non-trivial $\mu \in \mathcal{M}(E)$, the martingale limit

$$W_\infty^h(X) := \lim_{t \rightarrow +\infty} W_t^h(X) = \lim_{t \rightarrow +\infty} ce^{-\Lambda t} \sum_{i=1}^K u_i X_t^{(i)}$$

is non-degenerate if and only if $\Lambda > 0$ and

$$\sum_{i=1}^K u_i v_i \int_{(0, +\infty)} r \log^+(ru_i) \Pi^L(i, dr) + \sum_{i=1}^K \sum_{j=1}^K p_{ij} u_j v_i \int_{(0, +\infty)} r \log^+(r \sum_{k=1}^K p_{ik} u_k) \Pi^{NL}(i, dr) < +\infty.$$

Using elementary computation, one can reduce the above condition to

$$\int_{(0, +\infty)} r \log^+ r \Pi^L(i, dr) + \int_{(0, +\infty)} r \log^+ r \Pi^{NL}(i, dr) < +\infty \quad \text{for every } i \in E. \quad (7.3)$$

In particular, under condition (7.3), $P_\mu \left(\lim_{t \rightarrow +\infty} X_t^{(i)} = 0 \right) = 1$ for every $i \in E$ and every non-trivial $\mu \in \mathcal{M}(E)$ if and only if $\Lambda \leq 0$. This result coincides with [29, Theorem 6].

Now we give some other examples.

Example 7.2. Suppose that E is a bounded C^3 domain in \mathbb{R}^d ($d \geq 1$), m is the Lebesgue measure on E and that $\xi = (\xi_t, \Pi_x)$ is the killed Brownian motion in E . Suppose that ϕ^L and ϕ^{NL} are as given in Subsection 2.1. We assume Assumption 0 holds. We further assume that the probability kernel $\pi(x, dy)$ has a bounded density with respect to the Lebesgue measure m , i.e., $\pi(x, dy) = \pi(x, y) dy$ with $\pi(x, y)$ being bounded on $E \times E$. Assumption 1 and Assumption 3.(i) are trivially satisfied. Let $(\mathfrak{P}_t)_{t \geq 0}$ be the semigroup on $\mathcal{B}_b(E)$ uniquely determined by the integral equation (2.6). It follows from [24, Theorem] that Assumption 2, Assumption 3.(ii) are satisfied,

and that $(\mathfrak{P}_t)_{t \geq 0}$ is *uniformly primitive* in the sense of [24]. Thus for every $t > 0$, $f \in \mathcal{B}_b^+(E)$ and $x \in E$,

$$\left| \mathfrak{P}_t f(x) - e^{-\lambda_1 t} (f, \hat{h}) h(x) \right| \leq c_t e^{-\lambda_1 t} (f, \hat{h}) h(x), \quad (7.4)$$

where $c_t \geq 0$ satisfying $c_t \downarrow 0$ as $t \uparrow +\infty$, λ_1 is the constant in Assumption 2, and h, \hat{h} are the functions in Assumption 2. Let $\tilde{P}_t f(x) := e^{\lambda_1 t} h(x)^{-1} \mathfrak{P}_t(fh)(x)$ for $f \in \mathcal{B}^+(E)$, $t \geq 0$ and $x \in E$. Let $\tilde{p}(t, x, y)$ be the density of \tilde{P}_t with respect to the measure $\rho(dy) := h(y)\hat{h}(y)dy$ on E . By (7.4), we have for every $t > 0$, $f \in \mathcal{B}_b^+(E)$ and $x \in E$,

$$\left| \tilde{P}_t f(x) - \langle f, \rho \rangle \right| = \left| \int_E (\tilde{p}(t, x, y) - 1) f(y) \rho(dy) \right| \leq c_t \langle f, \rho \rangle.$$

It follows from this that

$$\sup_{x \in E} \operatorname{esssup}_{y \in E} |\tilde{p}(t, x, y) - 1| \leq c_t \rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

Hence Assumption 4 is satisfied. Assumption 3.(iii) will be satisfied if the function $\pi(x, y)$ satisfies

$$\int_E \pi(x, y) h(y) dy \leq ch(x) \quad \forall x \in \{z \in E : \gamma(z) > 0\}$$

for some constant $c > 0$, where h is the function in Assumption 2 and $\gamma(z)$ is as given in Subsection 2.1.

Example 7.3. Suppose that E is a bounded $C^{1,1}$ open set in \mathbb{R}^d ($d \geq 1$), m is the Lebesgue measure on E , $\alpha \in (0, 2)$, $\beta \in [0, \alpha \wedge d)$ and that $\xi = (\xi_t, \Pi_x)$ is an m -symmetric Hunt process on E satisfying the following conditions: (1) ξ has a Lévy system (N, t) where $N = N(x, dy)$ is a kernel given by

$$N(x, dy) = \frac{C_1}{|x - y|^{d+\alpha}} dy \quad x, y \in E$$

for some constant $C_1 > 0$. That is, for any $x \in E$, any non-negative measurable function f on $[0, +\infty) \times E \times E$ vanishing on $\{(s, y, y) : y \in E, s \geq 0\}$ and any stopping time T (with respect to the filtration of ξ),

$$\Pi_x \left[\sum_{s \leq T} f(s, \xi_{s-}, \xi_s) \right] = \Pi_x \left[\int_0^T \int_E f(s, \xi_s, y) N(\xi_s, dy) ds \right]. \quad (7.5)$$

(2) ξ admits a jointly continuous transition density $p(t, x, y)$ with respect to the Lebesgue measure and that there exists a constant $C_2 > 1$ such that

$$C_2^{-1} q_\beta(t, x, y) \leq p(t, x, y) \leq C_2 q_\beta(t, x, y) \quad \forall (t, x, y) \in (0, 1] \times E \times E,$$

where

$$q_\beta(t, x, y) = \left(1 \wedge \frac{\delta_E(x)}{t^{1/\alpha}} \right)^\beta \left(1 \wedge \frac{\delta_E(y)}{t^{1/\alpha}} \right)^\beta \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right). \quad (7.6)$$

Here $\delta_E(x)$ stands for the Euclidean distance between x and the boundary of E . Suppose that ϕ^L and ϕ^{NL} are as given in Subsection 2.1. We assume Assumption 0 holds. We further assume

that the probability kernel $\pi(x, dy)$ has a density $\pi(x, y)$ with respect to the Lebesgue measure m satisfying the condition

$$\pi(x, y) \leq C_3 |x - y|^{\epsilon - d} \quad \forall x, y \in E$$

for some positive constants C_3 and ϵ . In this case, Assumption 1 and Assumption 3.(i) are trivially satisfied. Let $\gamma(x, y) := \gamma(x)\pi(x, y)$ where $\gamma(x)$ is a bounded function as given in Subsection 2.1. Define

$$F(x, y) := \log \left(1 + C_1^{-1} |x - y|^{d+\alpha} \gamma(x, y) \right) \quad \forall x, y \in E.$$

It is obvious that there exists $C_4 > 0$ such that

$$0 \leq F(x, y) \leq C_4 (|x - y|^{\epsilon+\alpha} \wedge 1) \quad \forall x, y \in E,$$

and thus, by [9, Proposition 4.2], F belongs to the Kato class $\mathbf{J}_{\alpha, \beta}$ defined in [9]. The measure $\mu(dx) := -a(x)dx$ obviously belongs to the Kato class $\mathbf{K}_{\alpha, \beta}$ defined in [9] since a is a bounded function. For $0 < s \leq t < +\infty$, let $A_{s,t} := -\int_s^t a(\xi_r)dr + \sum_{s < r \leq t} F(\xi_{r-}, \xi_r)$. Let $(T_t)_{t \geq 0}$ be the Feynman-Kac semigroup of ξ given by

$$T_t f(x) := \Pi_x [\exp(A_{0,t}) f(\xi_t)], \quad t \geq 0, x \in E, f \in \mathcal{B}^+(E).$$

Now it follows from [9, Theorem 1.3] that the semigroup $(T_t)_{t \geq 0}$ has a jointly continuous density $q(t, x, y)$ with respect to the Lebesgue measure and there exists a constant $C_5 > 1$ such that

$$C_5^{-1} q_\beta(t, x, y) \leq q(t, x, y) \leq C_5 q_\beta(t, x, y) \quad \forall (t, x, y) \in (0, 1] \times E \times E. \quad (7.7)$$

Let $(\hat{T}_t)_{t \geq 0}$ be the dual semigroup of $(T_t)_{t \geq 0}$. By (7.7), one can easily show that for any $f \in \mathcal{B}_b(E)$, $T_t f$ and $\hat{T}_t f$ are bounded continuous functions on E , that T_t and \hat{T}_t are bounded operators from $L^2(E, m)$ into $L^\infty(E, m)$, and that $(T_t)_{t \geq 0}$ and $(\hat{T}_t)_{t \geq 0}$ are strongly continuous semigroups on $L^2(E, m)$. Similar to [13, (2.6)], we have

$$\exp(A_{0,t}) - 1 = - \int_0^t \exp(A_{s,t}) a(\xi_s) ds + \sum_{s \leq t} \exp(A_{s,t}) (\exp(F(\xi_{s-}, \xi_s)) - 1).$$

Using this, the Markov property of ξ and (7.5), one can show that for any $f \in \mathcal{B}_b(E)$ and any $x \in E$,

$$T_t f(x) = \Pi_x [f(\xi_t)] - \Pi_x \left[\int_0^t a(\xi_s) T_{t-s} f(\xi_s) ds \right] + \Pi_x \left[\int_0^t \int_E T_{t-s} f(y) \gamma(\xi_s, y) dy ds \right].$$

This implies that $(T_t)_{t \geq 0}$ is the unique strongly continuous semigroup on $L^2(E, m)$ associated with the bilinear form $(\mathcal{Q}, \mathcal{F})$ where

$$\begin{aligned} \mathcal{Q}(u, v) &:= \mathcal{E}(u, v) + \int_E a(x) u(x) v(x) dx - \int_E \int_E u(y) v(x) \gamma(x, y) dy dx \\ &= \mathcal{E}(u, v) - \int_E u(x) v(x) \mu(dx) - \int_E \int_E u(y) v(x) \left(e^{F(x, y)} - 1 \right) N(x, dy) dx, \quad \forall u, v \in \mathcal{F}, \end{aligned}$$

and $(\mathcal{E}, \mathcal{F})$ is the Dirichlet form of ξ on $L^2(E, m)$. Let \mathbf{L} and $\widehat{\mathbf{L}}$ be the generators of $(T_t)_{t \geq 0}$ and $(\widehat{T}_t)_{t \geq 0}$ respectively. Let $\sigma(\mathbf{L})$ and $\sigma(\widehat{\mathbf{L}})$ denote the spectrum of \mathbf{L} and $\widehat{\mathbf{L}}$ respectively. It follows from (7.7) and Jentzsch's theorem ([33, Theorem V.6.6, p. 337]) that the common value $-\lambda_1 := \sup \operatorname{Re}(\sigma(\mathbf{L})) = \sup \operatorname{Re}(\sigma(\widehat{\mathbf{L}}))$ is an eigenvalue of multiplicity 1 for both \mathbf{L} and $\widehat{\mathbf{L}}$, and that an eigenfunction h of \mathbf{L} associated with $-\lambda_1$ is bounded continuous and can be chosen strictly positive on E and satisfies $\|h\|_{L^2(E, m)} = 1$, and that an eigenfunction \widehat{h} of $\widehat{\mathbf{L}}$ associated with $-\lambda_1$ is bounded continuous and can be chosen strictly positive on E and satisfies $(h, \widehat{h}) = 1$. Thus Assumption 2 and 3.(ii) are satisfied. It follows from (7.7) and the equations $e^{-\lambda_1 t} h = T_1 h$, $e^{-\lambda_1 t} \widehat{h} = \widehat{T}_1 h$ that there exists a constant $C_6 > 1$ such that

$$C_6^{-1} \delta_E(x)^\beta \leq h(x) \leq C_6 \delta_E(x)^\beta, \quad C_6^{-1} \delta_E(x)^\beta \leq \widehat{h}(x) \leq C_6 \delta_E(x)^\beta \quad \forall x \in E.$$

It follows from this, (7.7) and the semigroup property that the semigroups $(T_t)_{t \geq 0}$ and $(\widehat{T}_t)_{t \geq 0}$ are intrinsically ultracontractive. For the definition of intrinsic ultracontractivity, see [30]. Let $\widetilde{P}_t f(x) := e^{\lambda_1 t} h(x)^{-1} T_t(fh)(x)$ for $f \in \mathcal{B}^+(E)$, $t \geq 0$ and $x \in E$. Then \widetilde{P}_t admits a density $\widetilde{p}(t, x, y)$ with respect to the probability measure $h(y) \widehat{h}(y) dy$ which is related to $q(t, x, y)$ by

$$\widetilde{p}(t, x, y) = \frac{e^{\lambda_1 t} q(t, x, y)}{h(x) \widehat{h}(y)} \quad \forall (t, x, y) \in (0, +\infty) \times E \times E.$$

Now it follows from [30, Theorem 2.7] that Assumption 4 is satisfied. As in the previous example, Assumption 3.(iii) will be satisfied if the function $\pi(x, y)$ satisfies

$$\int_E \pi(x, y) h(y) dy \leq c h(x) \quad \forall x \in \{z \in E : \gamma(z) > 0\}$$

for some constant $c > 0$, where $\gamma(z)$ is as given in Subsection 2.1.

One concrete example of ξ is the killed symmetric α -stable process in E . In this case, (7.6) is satisfied with $\beta = \alpha/2$, a fact which was first proved in [7].

Another concrete example of ξ is the censored symmetric α -stable process in E introduced in [4] when $\alpha \in (1, 2)$. In this case, (7.6) is satisfied with $\beta = \alpha - 1$, a fact which was first proved in [8].

In fact, by using [9], one could also include the case when E is a d -set, $\alpha \in (0, 2)$ and ξ is an α -stable-like process in E introduced in [10]. We omit the details.

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